UNIVERSITY OF OTAGO EXAMINATIONS 2012

PHYSICS

PHSI 461

Linear Systems and Noise with Applications
Semester One

(TIME ALLOWED: 3 HOURS)

This examination paper comprises 8 pages

Candidates should answer questions as follows:

Answer FOUR Questions

The following material is provided:

See page 2.

Use of calculators:

No restriction on the model of calculator to be used, but no device with communication capability shall be accepted as a calculator.
(Calculators are subject to inspection by examiners.)

Candidates are permitted copies of:

Other Instructions:

DO NOT USE RED INK OR PENCIL.

A table of Fourier transforms and properties

Forward: $F(\nu) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{j}2\pi\nu t} dt$ Inverse: $f(t) = \int_{-\infty}^{\infty} F(\nu)\, \mathrm{e}^{\mathrm{j}2\pi\nu t} d\nu$

Some properties		
$F(t)$	$\leftrightarrow$	$f(-\nu)$
$f^*(t)$	$\leftrightarrow$	$F^*(-\nu)$
$f(at)$	$\leftrightarrow$	$\frac{1}{\lvert a\rvert} F\left(\frac{\nu}{a}\right)$
$f(t-t_0)$	$\leftrightarrow$	$\mathrm{e}^{-\mathrm{j}2\pi\nu t_0} F(\nu)$
$\mathrm{e}^{\mathrm{j}2\pi\nu_0 t} f(t)$	$\leftrightarrow$	$F(\nu-\nu_0)$
$\frac{d^n}{dt^n} f(t)$	$\leftrightarrow$	$(\mathrm{j}2\pi\nu)^n F(\nu)$
$-\mathrm{j}2\pi t f(t)$	$\leftrightarrow$	$\frac{dF(\nu)}{d\nu}$
$\int_{-\infty}^{t} f(\tau)\, d\tau$	$\leftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu} F(\nu) + \frac{1}{2} F(0)\,\delta(\nu)$
$(f*g)(t)$	$\leftrightarrow$	$F(\nu)\, G(\nu)$
$f(t)\, g(t)$	$\leftrightarrow$	$(F*G)(\nu)$

Some transform pairs		
$\delta(t)$	$\leftrightarrow$	1
$u(t)\, \mathrm{e}^{-at}$	$\leftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu + a}$
$u(t)$	$\leftrightarrow$	$\frac{1}{2}\delta(\nu) + \frac{1}{\mathrm{j}2\pi\nu}$
$\exp(\mathrm{j}2\pi\nu_0 t)$	$\leftrightarrow$	$\delta(\nu-\nu_0)$
$\cos(2\pi\nu_0 t)$	$\leftrightarrow$	$\frac{1}{2}[\delta(\nu-\nu_0) + \delta(\nu+\nu_0)]$
$\sin(2\pi\nu_0 t)$	$\leftrightarrow$	$\frac{\mathrm{j}}{2}[-\delta(\nu-\nu_0) + \delta(\nu+\nu_0)]$
$\Pi(t)$	$\leftrightarrow$	$\mathrm{sinc}(\nu)$
$\mathrm{sgn}(t)$	$\leftrightarrow$	$\frac{1}{\mathrm{j}\pi\nu}$
$\sum_{k=-\infty}^{\infty} \delta(t-kT)$	$\leftrightarrow$	$\frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\left(\nu - \frac{k}{T}\right)$
$\exp(-\pi t^2)$	$\leftrightarrow$	$\exp(-\pi\nu^2)$

DFT: $X[r] = \frac{1}{N}\sum_{k=0}^{N-1} x[k] \exp\left(-\frac{\mathrm{j}2\pi rk}{N}\right)$ IDFT: $x[k] = \sum_{r=0}^{N-1} X[r] \exp\left(\frac{\mathrm{j}2\pi rk}{N}\right)$

$$\text{Hilbert transform: } \hat{f}(t) = f(t) * \frac{1}{\pi t}$$

$$\text{Convolution integral: } (f*h)(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, \mathrm{d}\tau$$

The Laplace transform and properties

$$\text{The Laplace transform : } F_L(s) = \int_0^{\infty} f(t)\mathrm{e}^{-st}\, \mathrm{d}t$$

$$\text{Complex inversion formula: } f(t) = \frac{1}{2\pi \mathrm{j}} \int_{\sigma-\mathrm{j}\infty}^{\sigma+\mathrm{j}\infty} F_L(s)\mathrm{e}^{st} \mathrm{d}s$$

Derivatives: $f'(t) \leftrightarrow sF_L(s) - f(0)$, $f''(t) \leftrightarrow s^2 F_L(s) - sf(0) - f'(0)$

TURN OVER

1. (a) Give definitions for each of the following:
   (i) A linear system.
   (ii) A shift invariant system.
   (iii) The impulse response for a linear time invariant system.
   (iv) The transfer function for a linear time invariant system.

   In your answers use $f_1(t)$ and $f_2(t)$ to represent arbitrary inputs and $g_1(t)$ and $g_2(t)$ their respective outputs. Use c_1 and c_2 to denote arbitrary numbers and t_0 an arbitrary time value.

   (8 marks)

   (b) Sketch each of the following functions and give their Fourier transforms.
   (i) $\Pi(t)$
   (ii) $\Pi(4t)$
   (iii) $\int_{-\infty}^{\infty} \Pi(4\tau)\Pi(t-\tau)\, d\tau$

   (6 marks)

   (c) For a linear time invariant system, *derive* an expression that relates the input ($f(t)$), the output ($g(t)$), and the impulse response function ($h(t)$).

   (6 marks)

   (d) Either by using the properties of convolution or directly from the definition, show that:

   If
$$F(t) = \int_{-\infty}^{t} f(\tau)\, d\tau,$$
   then
$$(F * g)(t) = \int_{-\infty}^{t} (f * g)(\tau)\, d\tau\,.$$

   (5 marks)

2. (a) Using Parseval's theorem,

$$\int_{-\infty}^{\infty} h(\tau)r(\tau)\,d\tau = \int_{-\infty}^{\infty} H(s)R(-s)\,ds\,,$$

and the properties of the Fourier transform, show that the Fourier transform of

$$f(t)g(t) \quad \text{is} \quad \int_{-\infty}^{\infty} F(s)G(\nu - s)\,ds.$$

(5 marks)

(b) Describe how the Fourier transform and inverse Fourier transform are defined for generalized functions. Then, assuming that the Fourier transform properties of Page 2 have been established for test functions, use your definition to show the following.

(i) $\delta(t) \leftrightarrow 1$

(ii) $j2\pi t f(t) \leftrightarrow -\dfrac{d}{d\nu}F(\nu)$

(iii) $f(at) \leftrightarrow \dfrac{1}{|a|}F\left(\dfrac{\nu}{a}\right)$

(iv) $\exp(j2\pi\nu_0 t)f(t) \leftrightarrow F(\nu - \nu_0)$

(v) If f is odd then so is F.

[Here the Fourier transform of an arbitary generalized function $f(t)$ is denoted $F(\nu)$.]

(15 marks)

(c) The signum function is defined by,

$$\mathrm{sgn}(t) = \begin{cases} -1, & \text{for } t < 0 \\ 0, & \text{for } t = 0 \\ 1, & \text{for } t > 0. \end{cases}$$

and has the derivative,

$$\frac{d}{dt}\mathrm{sgn}(t) = 2\delta(t)\,.$$

(i) Using $S(\nu)$ to denote the Fourier transform of the signum function, use the results given in part (b) to show that $j2\pi\nu S(\nu) = 2$, and give an argument why $S(0) = 0$.

(3 marks)

(ii) Derive an expression for $S(\nu)$ and then, using this, find the Fourier transform of the unit step function.

(2 marks)

3. (a) (i) Using its action on test functions show that the derivative of the unit step is the delta function.

(3 marks)

(ii) Calculate the second time derivative of

$$u(t)\exp(-t)\,.$$

(3 marks)

(b) The vector $\mathbf{x}$, has elements $x[k]$ where k takes on values from $0, 1, \ldots, (N-1)$. Show that

$$\mathrm{IDFT}\{\mathrm{DFT}\{\mathbf{x}\}\} = \mathbf{x}$$

(10 marks)

(c) A signal is periodic and repeats every 0.5 seconds; one period is sampled at 48 kHz resulting in $N = 24\,000$ samples.

The results form a vector $y[k], k \in \{0, 1, \ldots, (N-1)\}$, which has discrete Fourier transform $Y[r], r \in \{0, 1, \ldots, (N-1)\}$.

Which frequency components of the original sample contribute to the value of $Y[4800]$? (Write your answer in the form $a + nb$ where a and b are in Hertz and n is used to denote an arbitrary integer.)

(3 marks)

(d) Sketch the following functions and then calculate their Fourier transforms.

(i) $\Pi(t)\cos(\pi t)$ (2 marks)

(ii) $|\cos(\pi t)|$ (4 marks)

Hint: The following Fourier transform relationship might be useful

$$\sum_{k=-\infty}^{\infty} \delta\left(t - kT\right) \leftrightarrow \frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\left(\nu - \frac{k}{T}\right)$$

4. (a) Consider a real signal $f(t)$ with Fourier transform $F(\nu)$. The analytic signal $f_a(t)$ is defined by

$$f_a(t) = 2\int_0^\infty F(\nu)\exp(j2\pi\nu t)\,\mathrm{d}\nu,$$

and can also be written as $f_a(t) = f(t) + j\hat{f}(t)$, where $\hat{f}(t)$ is the Hilbert transform of $f(t)$.

(i) Show that $F_a(\nu) = 2u(\nu)F(\nu)$, where $u(\nu)$ is the unit step function. (3 marks)

(ii) Using this result, show that the Hilbert transform $\hat{f}(t)$ is also given by the integral

$$\hat{f}(t) = \int_{-\infty}^{\infty} \frac{f(\tau)}{\pi(t-\tau)}\,\mathrm{d}\tau.$$

(4 marks)

(b) Consider the double-sideband (DSB) modulated signal $f(t) = m(t)\cos(2\pi\nu_\mathrm{c}t)$ where

$$m(t) = \mathrm{sinc}(\nu_\mathrm{m}t) = \frac{\sin(\pi\nu_\mathrm{m}t)}{\pi\nu_\mathrm{m}t},$$

with $0 < \nu_\mathrm{m} < \nu_\mathrm{c}$.

(i) Calculate and sketch $F(\nu)$ for the DSB modulated signal. (5 marks)
[Note: $\mathrm{sinc}(bt) \leftrightarrow \frac{1}{b}\Pi\left(\frac{\nu}{b}\right)$]

(ii) Hence, deduce $F_a(\nu)$ for the DSB modulated signal and, using the inverse Fourier transform, find $f_a(t)$. (4 marks)

(iii) From your result for the analytic signal $f_a(t)$, find the Hilbert transform $\hat{f}(t)$ and show that the envelope of the signal is $|\mathrm{sinc}(\nu_\mathrm{m}t)|$. (2 marks)

(c) Now consider the single-sideband (SSB) modulated signal

$$f(t) = \mathrm{Re}\{m_a(t)\exp(j2\pi\nu_\mathrm{c}t)\}$$

with $m(t)$ as above. Calculate and sketch $F(\nu)$ for the SSB modulated signal. (4 marks)
[Note: $g^*(t) \leftrightarrow G^*(-\nu)$]

(d) Describe briefly how the DSB and SSB modulated signals can be demodulated with a reconstructed carrier signal. (3 marks)

5. (a) Consider a deterministic signal $f(t)$. Define precisely (i.e., in mathematical terms) what it means for $f(t)$ to be:

   (i) a finite energy signal, and

   (ii) a finite power signal. (4 marks)

   (b) Define the energy and power autocorrelation functions of $f(t)$. (2 marks)

   (c) Let $f(t)$ be a finite energy signal with Fourier transform $F(\nu)$. By considering the passage of this signal through an ideal band-pass filter, followed by an energy measurement, show that $|F(\nu)|^2$ gives the energy spectral density. (6 marks)

   (d) Show that the energy spectral density is the Fourier transform of the energy autocorrelation function. (4 marks)

   (e) What is the corresponding result for a finite power signal? (1 mark)

   (f) A simple USB modulated signal takes the form

$$f(t) = A\cos[2\pi(\nu_c + \nu_m)t],$$

   where A is a constant, ν_c is the carrier frequency, and $\nu_m (\ll \nu_c)$ is the modulating frequency. Calculate the power autocorrelation function and power spectral density of $f(t)$. When compared to other modulation formats, what advantageous feature of USB modulation does your result for the power spectral density highlight? (8 marks)

   Note: $\cos(\theta)\cos(\phi) = \frac{1}{2}[\cos(\theta + \phi) + \cos(\theta - \phi)]$

6. (a) Consider the diffusion equation in one dimension,

$$\frac{\partial P(x,t)}{\partial t} = \frac{1}{2} D \frac{\partial^2 P(x,t)}{\partial x^2},$$

where D is a constant.

(i) Assuming $P(x = \pm\infty, t) = 0$ for all t, use Fourier and Laplace transforms to find $P(x,t)$ for a given value of $P(x, t = 0)$. Hence show that, for $P(x, t = 0) = \delta(x)$,

$$P(x,t) = \frac{1}{\sqrt{2\pi D t}} \exp\left(-\frac{x^2}{2Dt}\right).$$

(8 marks)

(ii) Assume that $P(x,t)$ represents the probability density for a particle undergoing Brownian motion in one dimension. Calculate the mean position $\langle x(t) \rangle$ and mean position-squared $\langle x(t)^2 \rangle$ of the particle – in particular, give expressions for these quantities in terms of integrals over $P(x,t)$ and evaluate these integrals. (4 marks)

Note: $\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2} dx = \sqrt{\pi/a}$

(b) Brownian motion of a particle in one dimension can also be described by the stochastic differential equation

$$dx(t) = \sqrt{D}\, dW(t). \tag{1}$$

(i) What is $dW(t)$? Give its properties. (3 marks)

(ii) Using this stochastic differential equation, derive and solve equations of motion for $\langle x(t) \rangle$ and $\langle x(t)^2 \rangle$, and hence confirm your results from part (iii) above for a particle with initial position $x(t = 0) = 0$. (4 marks)

(iii) Repeat your calculation for the modified stochastic differential equation

$$dx(t) = -Ax(t)dt + \sqrt{D}\, dW(t), \tag{2}$$

where $A > 0$ is a constant. (4 marks)

(iv) Which of equations (1) and (2) above describe a *stationary process*? Explain. (Hint: Consider the limit $t \to \infty$.) (2 marks)

END