UNIVERSITY OF OTAGO EXAMINATIONS 2011

PHYSICS

PHSI 461

Linear Systems and Noise with Applications

Semester One

(TIME ALLOWED: 3 HOURS)

This examination paper comprises 8 pages

Candidates should answer questions as follows:

Answer FOUR Questions

The following material is provided:

See page 2.

Use of calculators:

No restriction on the model of calculator to be used, but no device with communication capability shall be accepted as a calculator.
(Calculators are subject to inspection by examiners.)

Candidates are permitted copies of:

Other Instructions:

DO NOT USE RED INK OR PENCIL.

TURN OVER

A table of Fourier transforms and properties

Forward: $F(\nu) = \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-\mathrm{j}2\pi\nu t}dt$ Inverse: $f(t) = \int_{-\infty}^{\infty} F(\nu)\,\mathrm{e}^{\mathrm{j}2\pi\nu t}d\nu$

Some properties

$$\begin{aligned}
F(t) &\leftrightarrow f(-\nu) \\
f^*(t) &\leftrightarrow F^*(-\nu) \\
f(at) &\leftrightarrow \tfrac{1}{|a|}F\left(\tfrac{\nu}{a}\right) \\
f(t-t_0) &\leftrightarrow \mathrm{e}^{-\mathrm{j}2\pi\nu t_0}F(\nu) \\
\mathrm{e}^{\mathrm{j}2\pi\nu_0 t}f(t) &\leftrightarrow F(\nu-\nu_0) \\
\tfrac{d^n}{dt^n}f(t) &\leftrightarrow (\mathrm{j}2\pi\nu)^n F(\nu) \\
-\mathrm{j}2\pi t f(t) &\leftrightarrow \tfrac{dF(\nu)}{d\nu} \\
\textstyle\int_{-\infty}^{t} f(\tau)\,d\tau &\leftrightarrow \tfrac{1}{\mathrm{j}2\pi\nu}F(\nu) + \tfrac{1}{2}F(0)\,\delta(\nu) \\
(f*g)(t) &\leftrightarrow F(\nu)\,G(\nu) \\
f(t)\,g(t) &\leftrightarrow (F*G)(\nu)
\end{aligned}$$

Some transform pairs

$$\begin{aligned}
\delta(t) &\leftrightarrow 1 \\
u(t)\,\mathrm{e}^{-at} &\leftrightarrow \tfrac{1}{\mathrm{j}2\pi\nu+a} \\
u(t) &\leftrightarrow \tfrac{1}{2}\delta(\nu) + \tfrac{1}{\mathrm{j}2\pi\nu} \\
\exp(\mathrm{j}2\pi\nu_0 t) &\leftrightarrow \delta(\nu-\nu_0) \\
\cos(2\pi\nu_0 t) &\leftrightarrow \tfrac{1}{2}\left[\delta(\nu-\nu_0)+\delta(\nu+\nu_0)\right] \\
\sin(2\pi\nu_0 t) &\leftrightarrow \tfrac{\mathrm{j}}{2}\left[-\delta(\nu-\nu_0)+\delta(\nu+\nu_0)\right] \\
\Pi(t) &\leftrightarrow \mathrm{sinc}(\nu) \\
\mathrm{sgn}(t) &\leftrightarrow \tfrac{1}{\mathrm{j}\pi\nu} \\
\sum_{k=-\infty}^{\infty}\delta(t-kT) &\leftrightarrow \frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(\nu-\frac{k}{T}\right) \\
\exp(-\pi t^2) &\leftrightarrow \exp(-\pi\nu^2)
\end{aligned}$$

DFT: $X[r] = \dfrac{1}{N}\displaystyle\sum_{k=0}^{N-1} x[k]\exp\left(-\dfrac{\mathrm{j}2\pi rk}{N}\right)$ IDFT: $x[k] = \displaystyle\sum_{r=0}^{N-1} X[r]\exp\left(\dfrac{\mathrm{j}2\pi rk}{N}\right)$

Hilbert transform: $\hat{f}(t) = f(t) * \dfrac{1}{\pi t}$, $\hat{F}(\nu) = -\mathrm{j}\,\mathrm{sgn}(\nu)F(\nu)$

Convolution integral: $(f*h)(t) = \displaystyle\int_{-\infty}^{\infty} f(\tau)\,h(t-\tau)\,\mathrm{d}\tau$

The Laplace transform and properties

The Laplace transform : $F_L(s) = \displaystyle\int_0^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t$

Complex inversion formula: $f(t) = \dfrac{1}{2\pi\mathrm{j}}\displaystyle\int_{\sigma-\mathrm{j}\infty}^{\sigma+\mathrm{j}\infty} F_L(s)\mathrm{e}^{st}\mathrm{d}s$

Derivatives: $f'(t) \leftrightarrow sF_L(s) - f(0)$, $f''(t) \leftrightarrow s^2F_L(s) - sf(0) - f'(0)$

1. (a) Using $f_1(t)$ and $f_2(t)$ to denote arbitrary inputs and $g_1(t)$ and $g_2(t)$ to denote the respective outputs, describe what it means for a system to be:

   (i) Linear. (2 marks)
   (ii) Causal. (2 marks)
   (iii) Time invariant. (2 marks)

   (b) The function $f(t)$ has Fourier transform $F(\nu)$. From the definition of the Fourier transform derive expressions, in terms of $F(\nu)$ for

   (i) $\frac{d}{dt}f(t)$. (3 marks)
   (ii) $f(t-t_0)$. (3 marks)
   (iii) The real part of $f(t)$. (3 marks)

   (c) Calculate the convolution of $\mathrm{sinc}(at)$ and $\mathrm{sinc}(bt)$, where a and b are positive real numbers and $a > b$. (4 marks)

   (d) Show that:

   (i) $f * g = g * f$. (3 marks)
   (ii) $(f * g)' = f' * g = f * g'$. (3 marks)

2. (a) The function $f(t)$ has Fourier transform $F(\nu)$. Give an expression for

$$\int_{-\infty}^{\infty} t^2 |f(t)|^2 \, dt$$

in terms of $F(\nu)$. (6 marks)

(b) (i) Give a definition of stability for a system (in terms of its inputs and outputs). (2 marks)

(ii) Consider the linear time invariant system with impulse response

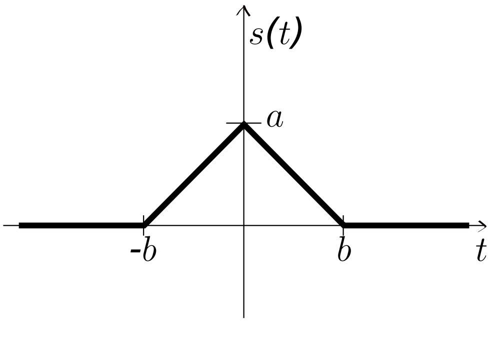
$$h(t) = u(t-1)\frac{1}{t}.$$

Here $u(t)$ is the unit step function. Is this system stable? Give a proof starting from the definition of stability that supports your answer.

(6 marks)

(c) The function $b(t)$ is periodic with period T. That is $b(t) = b(t + kT)$, for all t and for all integers k. Show that the Fourier transform of b consists of weighted delta functions at evenly spaced frequencies. What is the frequency spacing? (5 marks)

(d) What is the Fourier transform of $s(t)$, the function graphed below. Explain your answer.



(6 marks)

3. (a) Consider a linear time invariant system with impulse response $h(t)$. Show that the function $f(t) = \exp(j2\pi\nu t)$ is an eigenfunction. What is the eigenvalue? In light of this result explain why Fourier transforms are a useful tool when dealing with linear time invariant systems. (5 marks)

(b) Using λ for the wavelength, $k = 2\pi/\lambda$ and $f_0(x, y)$ for the wave amplitude in the xy-plane at $z = 0$, the paraxial diffraction integral states that

$$f_z(x,y) = \frac{1}{j\lambda z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_0(x_0,y_0)\exp\left\{\frac{jk}{2z}[(x-x_0)^2+(y-y_0)^2]\right\} dx_0\,dy_0.$$

The Fourier transform of this relationship is

$$F_z(u,v) = F_0(u,v)\exp\left[\frac{2\pi^2}{jk}(u^2+v^2)z\right].$$

(i) Consider an optical beam with a Gaussian amplitude distribution at $z = 0$,

$$f_0(x,y) = A_0\exp[-B_0(x^2+y^2)].$$

Show that the amplitude distribution remains Gaussian as it propagates with

$$f_z(x,y) = A(z)\exp[-B(z)(x^2+y^2)].$$

Include in your answer an expression for $B(z)$ (10 marks)

(ii) Suppose that paraxial beam has a wave amplitude in the xy-plane at $z = 0$ of

$$f_0(x,y) = q(x,y),$$

and that this leads to wave amplitude at $z = z_0$ of

$$f_{z_0}(x,y) = r(x,y).$$

If instead $f_0(x,y) = q(x,y)\exp(j\alpha x)$, what would be the resulting $f_{z_0}(x,y)$? (You can assume that α is small enough that the paraxial approximation remains valid.) (10 marks)

4. (a) Let $f(t)$ be a real-valued signal with Fourier transform $F(\nu)$.

   (i) Define the analytic signal $f_a(t)$ in terms of $F(\nu)$, and show that $F_a(\nu) = 2u(\nu)F(\nu)$, where $u(\nu)$ is the unit step function.

   (3 marks)

   (ii) Show that the real part of $f_a(t)$ is equal to $f(t)$, and define the Hilbert transform of $f(t)$, denoted by $\hat{f}(t)$, in terms of $f_a(t)$.

   (3 marks)

   (iii) If $m(t)$ is a real-valued, band-limited (i.e., $M(\nu) = 0$ for $|\nu| > \nu_m$) function and $\nu_\mathrm{c} > \nu_m$, show that the Hilbert transform of

$$f(t) = m(t)\cos(2\pi\nu_\mathrm{c}t)$$

   is

$$\hat{f}(t) = m(t)\sin(2\pi\nu_\mathrm{c}t).$$

   (8 marks)

   (b) A modulating signal $m(t) = \sin(2\pi\nu_\mathrm{m}t)$ is transmitted via a carrier of frequency ν_c ($\gg \nu_\mathrm{m}$) using upper sideband (USB) modulation.

   (i) Write down the expression for the transmitted signal $f(t)$ and sketch its spectrum $F(\nu)$.

   (4 marks)

   (ii) Show how the USB modulated signal can be demodulated using a reconstructed carrier signal and a low-pass filter.

   (4 marks)

   (iii) Discuss briefly some advantages of USB modulation in comparison with double sideband modulation and amplitude modulation.

   (3 marks)

Note:

$\cos(\theta)\cos(\phi) = \frac{1}{2}[\cos(\theta + \phi) + \cos(\theta - \phi)]$

$\sin(\theta)\sin(\phi) = \frac{1}{2}[\cos(\theta - \phi) - \cos(\theta + \phi)]$

$\sin(\theta)\cos(\phi) = \frac{1}{2}[\sin(\theta + \phi) + \sin(\theta - \phi)]$

5. (a) (i) Write down the definition of the two-sided Laplace transform $F_L(s)$ of the function $f(t)$ and describe what is meant by the *region of absolute convergence* of the Laplace transform.

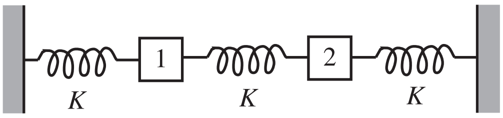
(3 marks)

(ii) Using the complex inversion formula, find the inverse Laplace transform (for $t > 0$ *and* $t < 0$) of

$$F_L(s) = \frac{1}{(s-1)(s^2+1)} \,,$$

where the region of convergence is $0 < \mathrm{Re}(s) < 1$. (8 marks)

(b) Consider the mass-spring system shown below. Two unit masses 1 and 2 are attached to three identical springs of force constant K.



If x and y are the displacements of masses 1 and 2 from their equilibrium positions, respectively, then the system is described by the equations of motion

$$\begin{aligned}\frac{d^2x}{dt^2} &= -Kx - K(x-y), \\ \frac{d^2y}{dt^2} &= -Ky + K(x-y).\end{aligned}$$

(i) Explain why the Laplace transforms $X_L(s)$ and $Y_L(s)$ of $x(t)$ and $y(t)$, respectively, are absolutely convergent in the region $\mathrm{Re}(s) > 0$.

(3 marks)

(ii) Setting $K = 1$, use Laplace transforms to find the solutions $x(t)$ and $y(t)$ of these equations if the masses are released *from rest* with each of the following initial displacements:

A. $x(0) = 1$, $y(0) = -1$,

B. $x(0) = 1$, $y(0) = 1$.

Hence identify the normal mode frequencies of the mass-spring system. [Hint: Consider the linear combinations $u(t) = x(t) + y(t)$ and $v(t) = x(t) - y(t)$.] (11 marks)

Note: The following Laplace transform pairs may be helpful.

$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2} \,, \quad \mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2} \qquad [\omega \text{ real},\ \mathrm{Re}(s) > 0]$$

6. (a) (i) A finite energy signal $f(t)$ has Fourier transform $F(\nu)$. By considering the passage of this signal through an ideal band-pass filter, followed by an energy measurement, show that $|F(\nu)|^2$ gives the energy spectral density.

(5 marks)

(ii) Define the *energy autocorrelation function* of the signal $f(t)$, and show that the energy spectral density is the Fourier transform of this function.

(6 marks)

(b) (i) The Ornstein-Uhlenbeck process is an example of a stochastic process and can be modelled by the *stochastic differential equation*

$$dx(t) = -\kappa x(t)dt + \sqrt{D}\,dW(t),$$

where κ and D are constants. Explain each of the terms in this equation and describe in particular the properties of the increment $dW(t)$.

(4 marks)

(ii) The stationary autocorrelation function for the Ornstein-Uhlenbeck process is

$$\langle x(t)x(s)\rangle = \frac{D}{2\kappa}\mathrm{e}^{-\kappa|t-s|}.$$

Define what is meant by "stationary process" and compute the power spectral density of the Ornstein-Uhlenbeck process.

(5 marks)

(iii) Give the autocorrelation function and power spectral density for a *white noise process.* Explain why such a process is an idealisation.

(5 marks)

END