UNIVERSITY OF OTAGO EXAMINATIONS 2010

PHYSICS

PHSI 461

Linear Systems and Noise with Applications
Semester One

(TIME ALLOWED: 3 HOURS)

This examination paper comprises 8 pages

Candidates should answer questions as follows:

Answer FOUR Questions

The following material is provided:

See page 2.

Use of calculators:

No restriction on the model of calculator to be used, but no device with communication capability shall be accepted as a calculator.
(Calculators are subject to inspection by examinors.)

Candidates are permitted copies of:

Other Instructions:

DO NOT USE RED INK OR PENCIL.
The values of PHYSICAL QUANTITIES which may be needed are given on page 2.
Symbols for physical quantities are given in italics.
Symbols for vector quantities are in bold.

A table of Fourier transforms and properties

Forward: $F(\nu) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{j}2\pi\nu t} dt$ Inverse: $f(t) = \int_{-\infty}^{\infty} F(\nu)\, \mathrm{e}^{\mathrm{j}2\pi\nu t} d\nu$

Some properties

$$\begin{aligned}
F(t) &\leftrightarrow f(-\nu) \\
f^*(t) &\leftrightarrow F^*(-\nu) \\
f(at) &\leftrightarrow \tfrac{1}{|a|} F\left(\tfrac{\nu}{a}\right) \\
f(t-t_0) &\leftrightarrow \mathrm{e}^{-\mathrm{j}2\pi\nu t_0} F(\nu) \\
\mathrm{e}^{\mathrm{j}2\pi\nu_0 t} f(t) &\leftrightarrow F(\nu-\nu_0) \\
\tfrac{d^n}{dt^n} f(t) &\leftrightarrow (\mathrm{j}2\pi\nu)^n F(\nu) \\
-\mathrm{j}2\pi t f(t) &\leftrightarrow \tfrac{dF(\nu)}{d\nu} \\
\int_{-\infty}^{t} f(\tau)\, d\tau &\leftrightarrow \tfrac{1}{\mathrm{j}2\pi\nu} F(\nu) + \tfrac{1}{2} F(0)\, \delta(\nu) \\
(f*g)(t) &\leftrightarrow F(\nu)\, G(\nu) \\
f(t)\, g(t) &\leftrightarrow (F*G)(\nu)
\end{aligned}$$

Some transform pairs

$$\begin{aligned}
\delta(t) &\leftrightarrow 1 \\
u(t)\, \mathrm{e}^{-at} &\leftrightarrow \tfrac{1}{\mathrm{j}2\pi\nu + a} \\
u(t) &\leftrightarrow \tfrac{1}{2}\delta(\nu) + \tfrac{1}{\mathrm{j}2\pi\nu} \\
\exp(\mathrm{j}2\pi\nu_0 t) &\leftrightarrow \delta(\nu-\nu_0) \\
\cos(2\pi\nu_0 t) &\leftrightarrow \tfrac{1}{2}[\delta(\nu-\nu_0) + \delta(\nu+\nu_0)] \\
\sin(2\pi\nu_0 t) &\leftrightarrow \tfrac{\mathrm{j}}{2}[-\delta(\nu-\nu_0) + \delta(\nu+\nu_0)] \\
\Pi(t) &\leftrightarrow \mathrm{sinc}(\nu) \\
\mathrm{sgn}(t) &\leftrightarrow \tfrac{1}{\mathrm{j}\pi\nu} \\
\sum_{k=-\infty}^{\infty} \delta(t-kT) &\leftrightarrow \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(\nu - \frac{k}{T}\right) \\
\exp(-\pi t^2) &\leftrightarrow \exp(-\pi\nu^2)
\end{aligned}$$

DFT: $X[r] = \dfrac{1}{N} \sum_{k=0}^{N-1} x[k] \exp\left(-\dfrac{\mathrm{j}2\pi rk}{N}\right)$ IDFT: $x[k] = \sum_{r=0}^{N-1} X[r] \exp\left(\dfrac{\mathrm{j}2\pi rk}{N}\right)$

Hilbert transform: $\hat{f}(t) = f(t) * \dfrac{1}{\pi t}$, $\hat{F}(\nu) = -\mathrm{j}\, \mathrm{sgn}(\nu) F(\nu)$

Convolution integral: $(f*h)(t) = \displaystyle\int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, \mathrm{d}\tau$

Complex inversion formula: $f(t) = \dfrac{1}{2\pi\mathrm{j}} \displaystyle\int_{\sigma-\mathrm{j}\infty}^{\sigma+\mathrm{j}\infty} F_L(s) \mathrm{e}^{st} \mathrm{d}s$

TURN OVER

1. (a) Assuming that the Fourier transform of $f(t)$ is $F(\nu)$, derive expressions, in terms of $F(\nu)$, for the Fourier transforms of

    (i) $tf(t)$

    (ii) $f(-t)$

    (iii) $\mathrm{Im}(f(t))$

    (iv) $\int_{-\infty}^{\infty} f(\tau)f^*(\tau - t)\, d\tau$

    where Im(.) takes the imaginary part of its argument. (10 marks)

   (b) For the function $g(t) = u(t)\sin(10\pi t)\exp(-t)$ (where $u(t)$ is the unit step function),

    (i) Sketch $g(t)$.

    (ii) Derive an expression for $G(\nu)$, the Fourier transform of $g(t)$.

    (iii) Sketch the absolute value and argument of $G(\nu)$.

    (10 marks)

   (c) The impulse response function, $h(t)$, for a system is real valued. Show that the input $f(t) = \sin(2\pi\nu t)$ leads to the output

$$g(t) = |H(\nu)|\sin(2\pi\nu t + arg(H(\nu))).$$

    (5 marks)

2. (a) Give definitions for each of the following:

   (i) a linear system

   (ii) a shift invariant system

   (iii) the impulse response for a linear time invariant system

   (iv) the transfer function for a linear time invariant system

   In your answers use $f_1(t)$, $f_2(t)$ to represent arbitrary inputs and $g_1(t)$, $g_2(t)$ their respective outputs. Use c_1 and c_2 to denote arbitrary numbers and t_0 an arbitrary time value. (8 marks)

   (b) For a linear time invariant system *derive* an expression that relates the input ($f(t)$), output ($g(t)$) and impulse response function $h(t)$. (8 marks)

   (c) Show that $\exp(j2\pi\nu t)$ is an eigenfunction of a linear time invariant system. Derive an expression for the eigenvalue in terms of the impulse response. (4 marks)

   (d) A linear system adds a single echo to an input, meaning that the input ($f(t)$) and output ($g(t)$) are related via

$$g(t) = f(t) + f(t - T).$$

   (i) Give an expression and sketch the impulse response for this system.

   (ii) Give an expression and sketch the real and imaginary parts of the transfer function.

   (iii) Is the system causal? Explain.

   (iv) Is the system stable? Explain.

   (5 marks)

3. (a) Using their action on test functions, show that the derivative of the unit step is the delta function. (7 marks)

   (b) A function $f(t)$ is sampled in time every T resulting in the sampled function

$$f_s(t) = \sum_{k=-\infty}^{\infty} f(t)\delta(t - kT),$$

   with the Fourier transform pairs $f(t) \leftrightarrow F(\nu)$ and $f_s(t) \leftrightarrow F_s(\nu)$.

   (i) Write down an expression for the Fourier transform of the sampled function ($F_s(\nu)$) in terms of $F(\nu)$, the Fourier transform of $f(t)$.

   (ii) After making up a shape for $F(\nu)$ sketch both this and $F_s(\nu)$.

   (iii) Give a definition for the Nyquist frequency and explain what is needed, both for $F(\nu)$ and for the filter, for $f(t)$ to be able to be recovered from $f_s(t)$ via an ideal low pass filter.

   (iv) Suppose that $F(\nu)$ was zero except for a very narrow (compared to the Nyquist frequency) band around $\nu = 30/T$. In this situation, can $f(t)$ be recovered from $f_s(t)$ via a filter? If so, what is the impulse response for this filter?

(18 marks)

4. (a) Let $f(t)$ be a real-valued signal with Fourier transform $F(\nu)$.

   (i) Define the analytic signal $f_a(t)$ in terms of $F(\nu)$, and show that $F_a(\nu) = 2u(\nu)F(\nu)$, where $u(\nu)$ is the unit step function. (3 marks)

   (ii) Show that the real part of $f_a(t)$ is equal to $f(t)$, and define the Hilbert transform of $f(t)$, denoted by $\hat{f}(t)$, in terms of $f_a(t)$. (3 marks)

   (iii) Let $f(t) = A\sin(2\pi\nu_{\mathrm{m}}t)\cos(2\pi\nu_{\mathrm{c}}t)$ where $\nu_{\mathrm{c}} > \nu_{\mathrm{m}} > 0$. Using your definition in (i), calculate the analytic signal $f_a(t)$. Hence find the Hilbert transform $\hat{f}(t)$ of $f(t)$ and the envelope of $f(t)$. (6 marks)

   (b) A modulating signal $m(t) = \cos(2\pi\nu_m t)$ is transmitted using a carrier $A\cos(2\pi\nu_c t)$ of frequency $\nu_c \gg \nu_m > 0$ (where A is a real constant).

   (i) Write down expressions for the modulated signal $f(t)$ for the cases of:
   1. Amplitude Modulation (AM), and
   2. Upper Sideband (USB) modulation.
   [Note: The Hilbert transform of $\cos(\omega t)$ is $\sin(\omega t)$ (for $\omega > 0$).] (2 marks)

   (ii) Compute the time-averaged power of the AM signal and show that the fraction of the total power in the carrier is $2/3$. (4 marks)

   (iii) The *power autocorrelation function* of a (deterministic) signal $f(t)$ is defined to be

$$\phi_{ff}^{p}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f^*(t) f(t+\tau)\,\mathrm{d}t.$$

   Compute $\phi_{ff}^{p}(\tau)$ for the USB modulated signal from part (b)(i). (4 marks)

   (iv) State the Wiener-Khinchin Theorem and compute the *power spectral density* of the USB modulated signal. What advantage to using USB modulation rather than AM is highlighted by your results? (3 marks)

Note:

$$\cos(\theta)\cos(\phi) = \tfrac{1}{2}[\cos(\theta+\phi) + \cos(\theta-\phi)]$$
$$\sin(\theta)\sin(\phi) = \tfrac{1}{2}[\cos(\theta-\phi) - \cos(\theta+\phi)]$$
$$\sin(\theta)\cos(\phi) = \tfrac{1}{2}[\sin(\theta+\phi) + \sin(\theta-\phi)]$$

5. (a) Compute the one-sided Laplace transform of $f(t) = \sinh(at)$. What is the region of convergence? (3 marks)

(b) Using the complex inversion formula, compute the inverse two-sided Laplace transform (i.e., $f(t)$ for $t > 0$ and $t < 0$) of

$$F_L(s) = \frac{1}{(s+2)(s-3)}$$

for the regions of convergence

(i) $-2 < \mathrm{Re}(s) < 3$, and

(ii) $\mathrm{Re}(s) > 3$. (10 marks)

(c) The Ornstein-Uhlenbeck process generalises the diffusion equation to the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = A\frac{\partial}{\partial x}(xP) + \frac{1}{2}D\frac{\partial^2}{\partial x^2}P\,,$$

where A and D are positive constants, and $P(x,t)$ is the time-dependent probability density of random variable x, e.g., the position of a particle.

(i) Show that the joint Fourier-Laplace transform of $P(x,t)$ satisfies the equation

$$sP_{FL}(u,s) - P_F(u,0) = -Au\frac{\partial}{\partial u}P_{FL}(u,s) - 2\pi^2 Du^2 P_{FL}(u,s)\,.$$

(6 marks)

[Hint: Use integration by parts and assume $P(x = \pm\infty, t) = 0$.]

(ii) Hence demonstrate that for $t \to \infty$ the probability density tends to the stationary state

$$P(x,\infty) = \frac{1}{\sqrt{2\pi D/A}}\exp\left(-\frac{x^2}{D/A}\right).$$

(6 marks)

[Note: $f(x) = \exp(-ax^2) \leftrightarrow F(u) = \sqrt{\pi/a}\exp(-\pi^2u^2/a)$]

TURN OVER

6. (a) Starting from the wave equation for light,

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} ,$$

and assuming a monochromatic, coherent field of the form $\psi(x,y,z,t) = \psi(x,y,z,0)e^{-j\omega t}$, show how the paraxial diffraction integral

$$f_z(x,y) = \frac{1}{j\lambda z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_0(x_0,y_0)\,\exp\left\{\frac{jk}{2z}\left[(x-x_0)^2 + (y-y_0)^2\right]\right\}\,dx_0\,dy_0$$

may be derived. Define the meanings of all of the symbols used and state the approximations made. (12 marks)

(b) Consider a diffraction experiment in which a mask with amplitude transmission function $f_0(x_0, y_0)$, placed in the xy-plane at $z = 0$, is illuminated from one side by a plane wave of light and the image of the transmitted light is viewed on a screen at a distance z on the other side.

(i) Define what is meant by the *Fraunhofer approximation* to the diffraction integral and describe the circumstances under which it is valid. Show that in this approximation the intensity $|f_z(x,y)|^2$ at z is given by

$$|f_z(x,y)|^2 = \frac{1}{\lambda^2 z^2}\left|F_0\left(\frac{x}{z\lambda}, \frac{y}{z\lambda}\right)\right|^2 ,$$

where F_0 is the two-dimensional Fourier transform of f_0. (6 marks)

(ii) Using the Fraunhofer approximation, calculate the diffraction pattern for a double slit experiment, assuming parallel, infinitely-thin (in the x-direction) slits that are located at $x_0 = \pm d/2$ and extend in the y-direction from $y_0 = -L/2$ to $y_0 = L/2$. Hence, show that the first minimum of the diffraction pattern along the x-axis occurs at $x_{\min} = \lambda z/(2d)$. (7 marks)

[Note: $\Pi(y/L) \leftrightarrow L\mathrm{sinc}(vL)$]