THE UNIVERSITY OF AUCKLAND

FIRST SEMESTER, 2009
Campus: City

PHYSICS

Linear Systems

(Time allowed: THREE hours)

INSTRUCTIONS TO CANDIDATES

Answer any **FOUR** questions.
All questions are of equal value.

The otago paper PHSI461 and the Auckland University paper PHYSICS 701 were in 2009 (and still are) taught jointly using a video link.

The exams sat were (and still are) the same in both institutions.

As far as I am aware in 2009 a Otago "branded" exam paper was not produced.

-- Jevon Longdell, Physics Dept, March 2011.

A table of Fourier transforms and properties

Forward: $F(\nu) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{j}2\pi\nu t} dt$ Inverse: $f(t) = \int_{-\infty}^{\infty} F(\nu)\, \mathrm{e}^{\mathrm{j}2\pi\nu t} d\nu$

Some properties

$F(t)$	$\leftrightarrow$	$f(-\nu)$
$f^*(t)$	$\leftrightarrow$	$F^*(-\nu)$
$f(at)$	$\leftrightarrow$	$\frac{1}{\lvert a\rvert} F\left(\frac{\nu}{a}\right)$
$f(t-t_0)$	$\leftrightarrow$	$\mathrm{e}^{-\mathrm{j}2\pi\nu t_0} F(\nu)$
$\mathrm{e}^{\mathrm{j}2\pi\nu_0 t} f(t)$	$\leftrightarrow$	$F(\nu-\nu_0)$
$\frac{d^n}{dt^n} f(t)$	$\leftrightarrow$	$(\mathrm{j}2\pi\nu)^n F(\nu)$
$-\mathrm{j}2\pi t f(t)$	$\leftrightarrow$	$\frac{dF(\nu)}{d\nu}$
$\int_{-\infty}^{t} f(\tau)\, d\tau$	$\leftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu} F(\nu) + \frac{1}{2} F(0)\, \delta(\nu)$
$(f * g)(t)$	$\leftrightarrow$	$F(\nu)\, G(\nu)$
$f(t)\, g(t)$	$\leftrightarrow$	$(F * G)(\nu)$

Some transform pairs

$\delta(t)$	$\leftrightarrow$	1
$u(t)\, \mathrm{e}^{-at}$	$\leftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu + a}$
$u(t)$	$\leftrightarrow$	$\frac{1}{2}\delta(\nu) + \frac{1}{\mathrm{j}2\pi\nu}$
$\exp(\mathrm{j}2\pi\nu_0 t)$	$\leftrightarrow$	$\delta(\nu-\nu_0)$
$\cos(2\pi\nu_0 t)$	$\leftrightarrow$	$\frac{1}{2}\left[\delta(\nu-\nu_0) + \delta(\nu+\nu_0)\right]$
$\sin(2\pi\nu_0 t)$	$\leftrightarrow$	$\frac{\mathrm{j}}{2}\left[-\delta(\nu-\nu_0) + \delta(\nu+\nu_0)\right]$
$\Pi(t)$	$\leftrightarrow$	$\mathrm{sinc}(\nu)$
$\mathrm{sgn}(t)$	$\leftrightarrow$	$\frac{1}{\mathrm{j}\pi\nu}$
$\sum_{k=-\infty}^{\infty} \delta(t-kT)$	$\leftrightarrow$	$\frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\left(\nu - \frac{k}{T}\right)$
$\exp(-\pi t^2)$	$\leftrightarrow$	$\exp(-\pi\nu^2)$

DFT: $X[r] = \frac{1}{N}\sum_{k=0}^{N-1} x[k] \exp\left(-\frac{\mathrm{j}2\pi rk}{N}\right)$ IDFT: $x[k] = \sum_{r=0}^{N-1} X[r] \exp\left(\frac{\mathrm{j}2\pi rk}{N}\right)$

Hilbert transform: $\hat{f}(t) = f(t) * \frac{1}{\pi t}$, $\hat{F}(\nu) = -\mathrm{j}\,\mathrm{sgn}(\nu) F(\nu)$

Convolution integral: $(f * h)(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, \mathrm{d}\tau$

1. (a) Define what it means for a system to be *linear*, and to be *shift-invariant* (or time-invariant). (4 marks)

(b) Define the *impulse response* of a linear shift-invariant system, and show how the impulse response relates input and output of the system. (4 marks)

(c) Define the *transfer function* of a linear shift-invariant system, and give a formula relating the transfer function to the impulse response of the system. (4 marks)

(d) Define what is meant by the *stability* of a linear shift-invariant system and show that it is equivalent to the impulse response of the system being absolutely integrable. (6 marks)

(e) A system with an 'echo' may be represented by a linear-time invariant system with the impulse response

$$h(t) = \sum_{k=0}^{\infty} a^k \delta(t - kT)$$

where $0 < a < 1$.

(i) Is this system stable? (3 marks)

(ii) Show that the transfer function $H(\nu)$ satisfies $H(v) = H(\nu + n/T)$ for all integers n. (4 marks)

2. (a) The function $f(t)$ has a Fourier transform $F(\nu)$. Derive expressions, in terms of $F(\nu)$, for the Fourier transforms of

$$\text{(i)} \quad \frac{d}{dt} f(t), \qquad \text{(ii)} \quad f^*(t), \qquad \text{and (iii)} \quad \mathrm{Re}(f(t)),$$

where * denotes a complex conjugate and Re(.) takes the real part of its argument. (6 marks)

(b) Show that if $f(t) \leftrightarrow F(\nu)$ and $g(t) \leftrightarrow G(\nu)$ are Fourier transform pairs, then $(f * g)(t) \leftrightarrow F(\nu)G(\nu)$. (7 marks)

(c) Find and sketch the Fourier transform of each of the functions:

$$w(t) = \begin{cases} 1 + \cos\left(\frac{\pi t}{M}\right), & |t| < M \\ 0 & \text{otherwise} \end{cases}$$

$$x(t) = \begin{cases} 1 - t^2, & |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$y(t) = \begin{cases} (1 - |t|)^2, & |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$z(t) = \begin{cases} (1 - |t|)^3, & |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

You may assume the transform pairs given on page 2. (12 marks)

3. A function $f(t)$ is sampled with a uniform sampling interval of T seconds to give the sequence $\{f[k] : f[k] = f(kT)\}$, or equivalently the generalized function

$$f_s(t) = \sum_{k=-\infty}^{\infty} f[k]\delta(t - kT).$$

(a) What is the relationship between the Fourier transforms of $f(t)$ and $f_s(t)$? Illustrate your answer with diagrams as appropriate. (4 marks)

(b) Using the result of part (a), or otherwise, derive Nyquist's sampling theorem, i.e. the conditions under which $f(t)$ can be reconstructed from the sample values $\{f[k]\}$. For the case that those conditions are satisfied, write down (or derive) a relationship giving $f(t)$ in terms of $f[k]$. (6 marks)

(c) A simple, though approximate, method of reconstruction is to linearly interpolate between sample values, i.e.,

$$f_I(t) = f[k] + \alpha f[k+1] \quad \text{when } t = kT + \alpha T \text{ with } \alpha \in [0, 1].$$

Linear interpolation may be performed by a linear time-invariant system that produces output f_I for input f_s.

(i) Give the impulse response of the interpolating system, and sketch the impulse response. (4 marks)

(ii) Derive the transfer function of the interpolating system. (4 marks)

(iii) Using your result from part (c)(ii), or otherwise, give the relationship between the Fourier transforms of the interpolated function f_I, and the original function f. Illustrate your answer with diagrams as appropriate. Under what condition is the linearly interpolated function f_I a good approximation to the original function f? (7 marks)

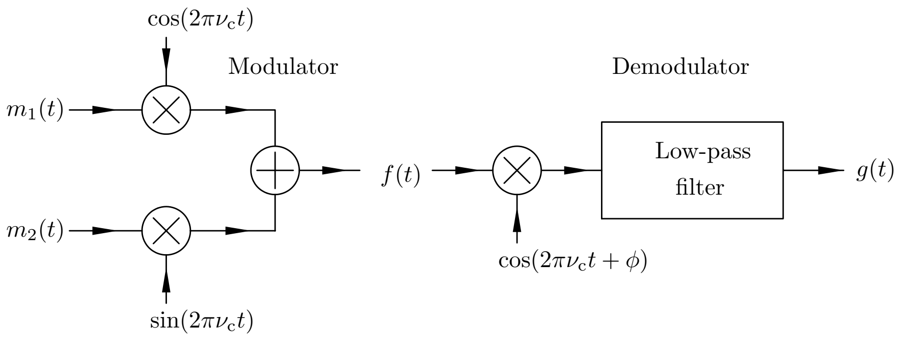
4. (a) A modulating signal $m(t) = \cos(2\pi\nu_{\rm m}t)$ is transmitted using a carrier of frequency $\nu_{\rm c}$ ($\nu_{\rm c} \gg \nu_{\rm m}$). Write down expressions for the modulated signal $f(t)$ and *sketch* its spectrum $F(\nu)$ for

(i) double sideband (DSB) modulation, and

(ii) upper sideband (USB) modulation. (6 marks)

Note: The Hilbert transform of $\cos(\omega t)$ is $\sin(\omega t)$ ($\omega > 0$).

(b) Two band-limited real signals $m_1(t)$ and $m_2(t)$ are applied to the quadrature modulator shown below, where the carrier frequency is $\nu_{\rm c}$. The spectra $M_1(\nu)$ and $M_2(\nu)$ vanish for $|\nu| \geq \nu_{\rm m}$.



(i) Write down expressions for the transmitted signal $f(t)$ and its spectrum $F(\nu)$. (3 marks)

(ii) At the receiver, the signal $f(t)$ is multiplied by a reconstructed carrier $\cos(2\pi\nu_{\rm c}t + \phi)$, and the output is low-pass filtered, so that only frequencies satisfying $|\nu| < \nu_{\rm f}$ are passed through. Calculate the output signal $g(t)$ for an arbitrary value of ϕ, and show that it is possible to recover $m_1(t)$ and $m_2(t)$ by making appropriate choices for ϕ. What relationships must hold between $\nu_{\rm m}$, $\nu_{\rm c}$, and $\nu_{\rm f}$ for this recovery to be possible? (6 marks)

Note: $\cos(\theta)\cos(\phi) = \frac{1}{2}[\cos(\theta+\phi) + \cos(\theta-\phi)]$ and $\sin(\theta)\cos(\phi) = \frac{1}{2}[\sin(\theta+\phi) + \sin(\theta-\phi)]$

(c) (i) A finite energy signal $f(t)$ has Fourier transform $F(\nu)$. By considering the passage of this signal through an ideal band-pass filter, followed by an energy measurement, show that $|F(\nu)|^2$ gives the energy spectral density. (5 marks)

(ii) Define the *energy autocorrelation function* of the signal $f(t)$, and show that the energy spectral density is the Fourier transform of this function. (5 marks)

5. (a) (i) Write down the definition of the two-sided Laplace transform $F_L(s)$ of the function $f(t)$ and describe what is meant by the *region of absolute convergence* of the Laplace transform. Write down the corresponding inverse Laplace transform relationship as a line integral in the complex plane. (5 marks)

(ii) Using the complex inversion formula, find the inverse Laplace transform of

$$F_L(s) = \frac{1}{(s-1)(s+2)}\,,$$

where the region of convergence is $-2 < \mathrm{Re}(s) < 1$. (8 marks)

(b) The coordinate $x(t)$ of a *driven damped harmonic oscillator* is described by the differential equation

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + \gamma \frac{\mathrm{d}x(t)}{\mathrm{d}t} + \omega^2 x(t) = f(t)\,,$$

where γ is the damping rate, $\omega/(2\pi)$ is the natural frequency of the oscillator, and $f(t)$ is the (time-dependent) driving force. Given the initial conditions

$$x(t=0) = 0\,, \qquad \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=0} = 0\,,$$

use Laplace transforms to show that

$$x(t) = \frac{1}{p_+ - p_-}\int_0^t \left[\mathrm{e}^{p_+(t-\tau)} - \mathrm{e}^{p_-(t-\tau)}\right] f(\tau)\,\mathrm{d}\tau\,,$$

where

$$p_\pm = -\frac{\gamma}{2} \pm \frac{1}{2}\sqrt{\gamma^2 - 4\omega^2}\,.$$

(12 marks)

6. (a) A plane wave of light of wavelength λ and unit amplitude is normally incident on a screen which is opaque except for a circular aperture of radius a.

(i) Use the paraxial diffraction integral to write down an expression for the amplitude of the light at a point (x_P, y_P) on a plane located at distance d on the other side of the screen. (Note: You do not have to evaluate the integral.) (2 marks)

(ii) Specialise now to the point P lying on the z-axis (i.e., $x_P = y_P = 0$). Evaluate the integral (using polar coordinates) and calculate the intensity. Show that the intensity at P is a minimum when $a^2 = 2n\lambda d$, where n is an integer. (9 marks)

Note: The paraxial diffraction integral is

$$f_z(x, y) = \frac{1}{j\lambda z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_0(x_0, y_0) \exp\left\{ \frac{jk}{2z} \left[(x - x_0)^2 + (y - y_0)^2 \right] \right\} \mathrm{d}x_0 \, \mathrm{d}y_0 \,,$$

where $k = 2\pi/\lambda$.

(b) Describe the *Fraunhofer approximation* to diffraction from an aperture. What is its region of validity for the circular aperture described above? (5 marks)

(c) Consider a Gaussian beam, which at $z = 0$ is described by $f_0(x, y) = A_0 \exp[-B_0(x^2 + y^2)]$, where A_0 and B_0 are *real.* In the paraxial approximation, show that the beam remains Gaussian as it propagates, but is described by the form $f_z(x, y) = A(z) \exp[-B(z)(x^2 + y^2)]$, where $A(z)$ and $B(z)$ are complex for $z \neq 0$. Give the expressions for $A(z)$ and $B(z)$ and show that the *half-width* of the beam, $w(z)$, is given by

$$w(z) = \sqrt{\frac{1 + 4B_0^2 z^2/k^2}{B_0}} \,.$$

(9 marks)

Note: The Fourier transform of the paraxial diffraction integral is

$$F_z(u, v) = F_0(u, v) \exp\left[-j\frac{2\pi^2}{k}(u^2 + v^2)z \right] ,$$

and $\exp(-ax^2) \leftrightarrow \sqrt{\pi/a}\, \exp(-\pi^2 u^2/a)$.