UNIVERSITY OF OTAGO EXAMINATIONS 2008

PHYSICS

PHSI 461

Communications and Noise

Semester One

(TIME ALLOWED: 2 HOURS)

This examination paper comprises 5 pages

Candidates should answer questions as follows:

Answer THREE QUESTIONS.
All questions carry equal weight.

The following material is provided:

A table of useful formulas is given on page 2.

Use of calculators:

No restriction on the model of calculator to be used, but no device with communication capability shall be accepted as a calculator.
(Calculators are subject to inspection by examiners.)

Candidates are permitted copies of:

Other Instructions:

DO NOT USE RED INK OR PENCIL.

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UNIVERSITY OF OTAGO PHYSICS DEPARTMENT
USEFUL FORMULAS
(for use in examinations)

A table of Fourier transforms and properties

Forward: $F(\nu) = \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-\mathrm{j}2\pi\nu t} dt$ Inverse: $f(t) = \int_{-\infty}^{\infty} F(\nu)\,\mathrm{e}^{\mathrm{j}2\pi\nu t} d\nu$

Some properties

$F(t)$	$\leftrightarrow$	$f(-\nu)$
$f^*(t)$	$\leftrightarrow$	$F^*(-\nu)$
$f(at)$	$\leftrightarrow$	$\frac{1}{\lvert a\rvert}F\left(\frac{\nu}{a}\right)$
$f(t-t_0)$	$\leftrightarrow$	$\mathrm{e}^{-\mathrm{j}2\pi\nu t_0}F(\nu)$
$\mathrm{e}^{\mathrm{j}2\pi\nu_0 t}f(t)$	$\leftrightarrow$	$F(\nu-\nu_0)$
$\frac{d^n}{dt^n}f(t)$	$\leftrightarrow$	$(\mathrm{j}2\pi\nu)^n F(\nu)$
$\int_{-\infty}^{t} f(\tau)\,d\tau$	$\leftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu}F(\nu) + \frac{1}{2}F(0)\,\delta(\nu)$
$(f*g)(t)$	$\leftrightarrow$	$F(\nu)\,G(\nu)$
$f(t)\,g(t)$	$\leftrightarrow$	$(F*G)(\nu)$

Some transform pairs

$\delta(t)$	$\leftrightarrow$	1
$u(t)\,\mathrm{e}^{-at}$	$\leftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu+a}$
$u(t)$	$\leftrightarrow$	$\frac{1}{2}\delta(\nu) + \frac{1}{\mathrm{j}2\pi\nu}$
$\exp(\mathrm{j}2\pi\nu_0 t)$	$\leftrightarrow$	$\delta(\nu-\nu_0)$
$\cos(2\pi\nu_0 t)$	$\leftrightarrow$	$\frac{1}{2}[\delta(\nu-\nu_0)+\delta(\nu+\nu_0)]$
$\sin(2\pi\nu_0 t)$	$\leftrightarrow$	$\frac{\mathrm{j}}{2}[-\delta(\nu-\nu_0)+\delta(\nu+\nu_0)]$
$\Pi(t)$	$\leftrightarrow$	$\mathrm{sinc}(\nu)$
$\mathrm{sgn}(t)$	$\leftrightarrow$	$\frac{1}{\mathrm{j}\pi\nu}$
$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\leftrightarrow$	$\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(\nu-\frac{k}{T}\right)$
$\exp\left(-\pi t^2\right)$	$\leftrightarrow$	$\exp\left(-\pi\nu^2\right)$

DFT: $X[r] = \frac{1}{N}\sum_{k=0}^{N-1} x[k]\exp\left(-\frac{\mathrm{j}2\pi rk}{N}\right)$ IDFT: $x[k] = \sum_{r=0}^{N-1} X[r]\exp\left(\frac{\mathrm{j}2\pi rk}{N}\right)$

Hilbert transform: $\hat{f}(t) = f(t) * \frac{1}{\pi t}$, $\hat{F}(\nu) = -\mathrm{j}\,\mathrm{sgn}(\nu)F(\nu)$

Convolution integral: $(f*h)(t) = \int_{-\infty}^{\infty} f(\tau)\,h(t-\tau)\,\mathrm{d}\tau$

1. (a) Let $\hat{f}$ and $\hat{g}$ denote the Hilbert transforms of functions f and g, respectively.
   (i) Give Rayleigh's theorem for energy invariance and use it to show that
$$\int_{-\infty}^{\infty} |f(t)|^2 = \int_{-\infty}^{\infty} \left|\hat{f}(t)\right|^2 .$$
   (ii) Show that
$$f * g = -\hat{f} * \hat{g}.$$
   (iii) Define the energy autocorrelation of a function $f(t)$. Using the Wiener-Khinchin theorem, or otherwise, show that the energy autocorrelation of f equals the energy autocorrelation of $\hat{f}$.

   (b) Let $f(t)$ be a causal function with no singularity at $t = 0$.
   (i) Derive the two Kramers-Kronig relationships which relate the real and imaginary parts of $F(\nu)$.
   (ii) Show that $f(t) = 2u(t)f_h(t)$ where $f_h(t) = [f(t) + f^*(-t)]/2$ is a Hermitian function. Show how the Fourier transform of $f_h(t)$ is simply related to $F(\nu)$.
   (iii) If the real part of the Fourier transform of $f(t)$ is
$$\text{Re}[F(\nu)] = \begin{cases} 1 & \text{for } |\nu| < 1, \\ 0 & \text{otherwise} \end{cases}$$
   find $\text{Im}[F(\nu)]$ and $f(t)$ and sketch their graphs.

2. (a) If $F(\nu)$ is the Fourier transform of $f(t)$, derive the Fourier transform of
   (i) $\dfrac{d}{dt}f(t)$,
   (ii) $f^*(t)$,
   where $f^*(t)$ is the complex conjugate of $f(t)$.

   (b) The function $f(t) = f_1(t) + jf_2(t)$, where both $f_1(t)$ and $f_2(t)$ are real, has Fourier transform $F(\nu)$. Find the Fourier transforms of $f_1(t)$ and $f_2(t)$ separately in terms of $F(\nu)$.

   (c) Consider the function
$$f(t) = \begin{cases} 1 & \text{for } \cos(2\pi\nu_0 t) > 0, \\ 0 & \text{otherwise.} \end{cases}$$
   (i) Sketch the function $f(t)$, find the Fourier transform of $f(t)$ and sketch the Fourier transform.
   (ii) Find and sketch the Fourier transform of $m(t)f(t)$ where $m(t)$ has bandwidth less than $\nu_0/2$.
   (iii) Specify the cutoff frequency and pass-band gain of a low-pass filter that gives output $m(t)$ for input $m(t)f(t)$.

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3. This question relates to the Klein-Gordon equation in one space dimension

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a\frac{\partial^2 u(x,t)}{\partial x^2} - bu(x,t)$$

where a and b are positive constants. The Klein–Gordon equation is the relativistic version of the Schrödinger equation.

(a) Verify that the Klein-Gordon equation in one space dimension has travelling wave solutions

$$\exp\{j2\pi s[x - c(s)t]\} \text{ and } \exp\{j2\pi s[x + c(s)t]\}$$

and give the phase speed $c(s)$ as a function of spatial frequency s.

(b) Derive the relationship between wave-number k and radial frequency ω, and hence find the group velocity of a wave packet centred at frequency ν_0.
Note: The wave-number $k \equiv 2\pi s$ where s is the spatial frequency, and radial frequency $\omega \equiv 2\pi\nu$ for frequency ν.

(c) Consider the initial value problem for $u(x,t)$, $t > 0$, consisting of the Klein-Gordon equation with initial position $u(x,0) = v(x)$ specified and initial derivative $\frac{\partial}{\partial t}u(x,0) = w(x)$ specified.

(i) By taking the Fourier transform of the initial value problem, in the x-direction, give an initial value problem for each spatial frequency s.

(ii) Give a linear combination of the travelling wave solutions that solve the transformed initial value problem for general initial value $v(x)$ and zero initial derivative, i.e., $w(x) = 0$. Hence give the transfer function from initial value to solution $u(x,t_0)$ for $t_0 > 0$.

(iii) Give a linear combination of the travelling wave solutions that solve the transformed initial value problem for zero initial value, i.e., $v(x) = 0$, and general initial derivative $w(x)$. Hence give the transfer function from initial derivative to solution $u(x,t_0)$ for $t_0 > 0$.

(iv) By writing the inverse transform of the sum of the solutions to the initial value problems in parts (ii) and (iii), give an expression for the solution $u(x,t)$ for $t > 0$ to the initial value problem for general v and w. You do not need to evaluate the inverse transform.

4. A light beam propagating nearly parallel to the z-axis is described by field components

$$\psi(x, y, z, t) = f_z(x, y) \exp\{\mathrm{j}(kz - \omega t)\}.$$

In the paraxial approximation of Fourier optics, propagation of the beam through a distance d is described by convolving the amplitude $f_z(x, y)$ with

$$h(x, y) = \frac{1}{\mathrm{j}\lambda d} \exp\left\{\frac{\mathrm{j}k}{2d}\left(x^2 + y^2\right)\right\}$$

while propagation through a thin lens of focal length f is described by multiplying the amplitude by

$$\exp\left\{-\frac{\mathrm{j}k}{2f}\left(x^2 + y^2\right)\right\}.$$

(a) A transparency in the plane $z = 0$ is illuminated with plane, monochromatic light so that $f_0(x, y)$ is known. A thin lens of focal length f is placed in the plane $z = f$ and a screen is placed in the plane $z = 2f$. Show that the amplitude of the light on the screen is related simply to the two-dimensional Fourier transform of the amplitude $f_0(x, y)$ in the plane $z = 0$.

(b) Consider the same situation as above, except that the lens (of focal length f) is now located in the plane $z = 2f$, while the screen is located at $z = 4f$. Relate the amplitude distribution on the screen to the amplitude distribution at $z = 0$. *Note:* You may find it useful to use the result

$$\int_{-\infty}^{\infty} \exp(\mathrm{j}\alpha x)\, dx = 2\pi\delta(\alpha).$$

(c) A Gaussian beam of the form $f_z(x, y) = A \exp\{-B(x^2 + y^2)\}$ propagates along the z axis and is incident at its waist on a thin convex lens of focal length f located in the plane $z = 0$. The half-width of the beam at its waist is w_0. Find the position of the waist and the half-width of the beam at the waist after it has passed through the lens. What happens to the position of the waist as $w_0 \to \infty$?

Note: For a Gaussian beam propagating a distance d through free space the parameter B changes according to

$$\frac{1}{B(d)} = \frac{1}{B(0)} + \frac{2\mathrm{j}d}{k}.$$