Decay of a dark soliton in a warm Bose-Einstein condensate: a variational approach

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Abstract

We present a variational approach describing the decay of a dark soliton in a one-dimensional warm Bose-Einstein condensate. At temperatures approaching the critical temperature, there is a significant thermal fraction and interactions between the condensate and thermal cloud can not be neglected. We use a projected stochastic Gross-Pitaevskii equation based on c-field techniques to account for these inherently stochastic interactions. We also undertake a numerical investigation of damping and the role of stochastic noise in dark soliton decay.

We treat the stochastic Gross-Pitaevskii equation as a perturbation of the Gross-Pitaevskii equation and find a perturbed Lagrangian which describes the interactions of the condensate and thermal cloud. This Lagrangian is based on microscopic theory and provides an accurate description of decay in a warm Bose-Einstein condensate. The first SGPE based Lagrangian variational damping solution is found for a single soliton in an infinite system. The solution shows good quantitative agreement with the numerical simulation of the Damped Gross-Pitaevskii equation. The single soliton lifetime is calculated as is the energy and momentum during the decay. Variational results for two damped, counter-propagating, widely separated, identical solitons in an infinite system are also obtained.

The numerical investigation of the role of noise shows a number of interesting results. Noise induced decoupling of solitons is witnessed in a periodic system. Noise induced decay of a stationary soliton in a finite homogenous system is observed. Most interestingly, we find that in contrast to vortex behaviour found by Rooney *et al.* [38], the averaged effects of thermal stochastic noise are to slow the soliton decay, relative to the predictions of the damped Gross-Pitaevskii equation. The thermal noise term causes a temperature dependent reduction in the damping rate. The averaged stochastic results approach the damping Gross-Pitaevskii equation in the limit of zero temperature.

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Chapter 1

Introduction

At zero temperature a Bose-Einstein condensate can be well described by the Gross-Pitaevskii equation (GPE), correctly predicting a number of experimental observations [18]. The one-dimensional GPE is a completely integrable nonlinear equation (see section 3.2.1). This means that among other things the GPE has exact solotonic solutions. These soliton solutions are stable localized translating waves which come in two forms: bright and dark. For an introduction to the properties of solitons refer to section 2. For a review of solitons in a BEC as well as some information on the formation and experimental setup see [7, 23].

This research project is concerned with dark solitons within the Gross-Pitaevskii framework [15]. This project deals with dark solitons from a theoretical point of view and compares analytic results to one dimensional numerical simulations. Dark solitons have been experimentally observed in BEC's, see for example [45]. Collisions between solitons have also been studied [41]. The study of the dynamics of dark solitons in a quasi one-dimensional BEC has been conducted for single solitons as well as multi solitons under the Gross-Pitaevskii framework [15].

The dynamics of solitons at low temperatures are well understood. However, the dynamics of dark solitons in a high temperature Bose-Einstein is a topic of much active research. In a high temperature Bose-Einstein condensate, where the temperature is larger than the single particle energy, there exists a significant thermal cloud. This thermal cloud is coupled to the condensate via scattering processes and is described by a stochastic theory. The resulting stochastic differential equation is known as the Stochastic Gross-Pitaevskii equation (SGPE) and arises from the c-field formalism [9]. The SGPE formalism has been presented in rigorous detail [25, 25], and gives a complete description of the high temperature open system. The GPE contains the two body interactions of the particles in the condensate and describes the mean field of the condensate at zero temperature. The damped GPE (DGPE) incorporates dissipation but neglects the stochastic effects of the thermal component. The SGPE contains the full effects of damping and stochastic thermal fluctuations. The soliton behaviour is different in each case, as described in figure 1.1.

It has been observed that at high temperatures solitons undergo anti-damping decay [15]. There has been some success in finding approximate analytic solutions for the dynamics of warm Bose-Einstein condensates via variational theories. Notably, both damping and noise were incorporated in the description of a gaussian condensate by Stoof

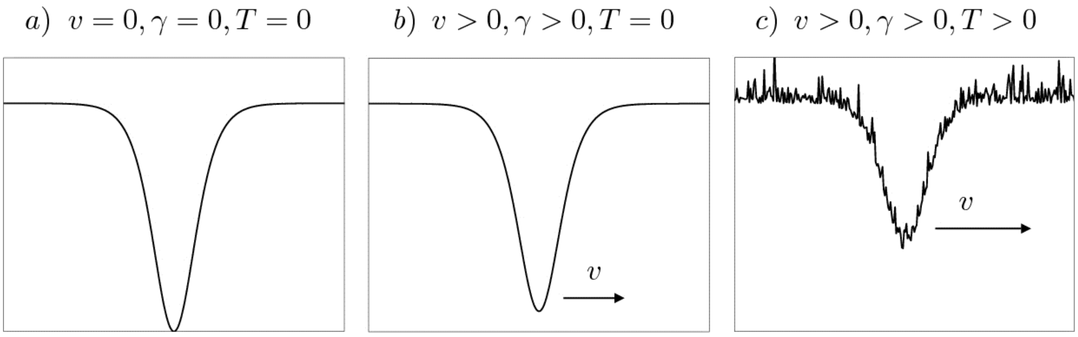

Figure 1.1: Comparison of soliton behaviour in the GPE, damped GPE and SGPE formalism. In the GPE description ($\gamma = T = 0$) the soliton will travel at constant velocity. An initially stationary soliton a) will remain stationary. In the damped GPE ($\gamma > 0, T = 0$) formalism b) a soliton will non-zero initial velocity will accelerate, the velocity will increase and the depth will decrease until the soliton disappears at $v = c$ the speed of sound in the condensate. In the full SGPE ($\gamma > 0, T > 0$) formalism c) the soliton wavefunction experiences damping and temperature dependent thermal fluctuations.

and Duine [20] using a stochastic differential equation which is equivalent to the SGPE formulation used here. Dark solitons exist within a background condensate, which makes them more difficult to describe and has prevented a full description of dark solitons using the SGPE formalism. Numerical approaches to the dynamics of Dark soliton dynamics in Bose-Einstein condensates at finite temperature have been considered, see [17, 30]. We wish to study the decay and dynamics of dark solitons within the Stochastic Gross-Pitaevskii framework, using a Lagrangian variational approach.

This research project attempts to describe the decay and dynamics of a dark soliton in a one-dimensional warm Bose-Einstein condensate (BEC). The aims of the project are:

1. Develop an analytical method for treating the decay process of a dark soliton in a one-dimensional finite temperature Bose-Einstein condensate, using the perturbed Lagrangian variational formulation.

2. Test this theory against numerical simulations of the decay process and carry out an investigation of the effects of thermal noise on soliton decay.

Chapter 2

Introduction to Solitons

Solitons are wave solutions of the Gross-Pitaevskii equation (or more generally the non-linear Schrödinger equation, see section 3.2.2). Solitons exist in a number of areas in physics, see [23] and references therein. Optical solitons have been observed as temporal pulses in optical fibers and spatial structures in bulk media and waveguides. Solitons can be observed as standing waves in a discrete mechanical system, high frequency waves in magnetic films and in a complex plasma. We will concern ourselves with dark solitons in Bose-Einstein condensates, although the variational techniques have been used in a number of soliton applications. Soliton waves and some of their unique properties were first discovered in 1844, by John Scott Russell. In his own words,

> "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation"[40].

Russell spent some time investigating these 'waves of translation' and found a number of properties;

1. The waveform is stable and can travel long distances without appreciable change.

2. The speed of the wave depends on its size and width.

3. The soliton waves don't merge, a smaller soliton will pass through a larger one.

4. If the wave amplitude becomes too large it will split into two solitons.

Since 1844, soliton solutions have been studied extensively in a number of contexts, and many of the properties that hold for solitons in a channel hold in these other systems. Solitons are travelling waves characterized by a localized constant wave form. While this is a well known result for the familiar wave equation, it is very rare for a non-linear equation. In general the non-linear effects cause the wave to 'bunch up' and dispersive effects cause the wave to spread out. In the case of the soliton these two effects exactly cancel. Furthermore if two solitons solution pass through each other they remain unchanged apart from a phase shift. Because of the constant localized wave form, solitons can be regarded as the nonlinear wave analog of a particle. In fact it is found that the soliton position (the density minima in the case of a dark soliton), satisfies the equation of motion of a particle in a potential, see section 5.2.1.

2.1 Dark Solitons in a BEC

In the context of Bose-Einstein condensates there are two distinct types of solutions. They are termed dark and light solitons. A light soliton is a localized density increase relative to the background condensate. A dark soliton is a localized density decrease. Physically the two different cases correspond to attractive and repulsive interaction potentials. The sign of the nonlinear term in the GPE is determined by the scattering length, a. Attractive potentials correspond to a negative scattering length, and repulsive potentials correspond to a positive scattering length. The scattering length is a property of the atoms used. ^{87}Rb and ^{23}Na have positive scattering lengths and permit dark soliton solutions. The general dark soliton solution (ignoring the rotating phase factor) of the one-dimensional GPE with no external potential $V(x) = 0$, is

$$\psi(x,t) = \sqrt{n}\left(i\frac{v}{c} + \sqrt{1-\frac{v^2}{c^2}}\tanh\left(\frac{x-vt}{\xi}\sqrt{1-\frac{v^2}{c^2}}\right)\right),$$

where the length scale of the soliton is set by the healing length ξ, defined as

$$\xi = \hbar/\sqrt{m\mu}.$$

The velocity determines both the amplitude of the density peak and the width. The depth of the soliton is largest for $v = 0$, a stationary 'black' soliton, where the density goes to zero at the centre. As the velocity increases the soliton decays, in the limit $v \to c$ the soliton disappears. The change of phase from $x = -\infty$ to $x = +\infty$ is given by

$$\Delta\theta = 2\cos^{-1}\left(\frac{v}{c}\right),$$

for $v = 0$, $\Delta\theta = \pi$ and as $v \to c$, $\Delta\theta \to 0$. In one dimension if two dark solitons of different velocity exist and are copropagating, they will move relative to each other and may pass through each other. When interference occurs the resulting waveform is not simply the superposition of the two individual solitons. However, when the two separated again the wave forms are unchanged except for a phase shift.

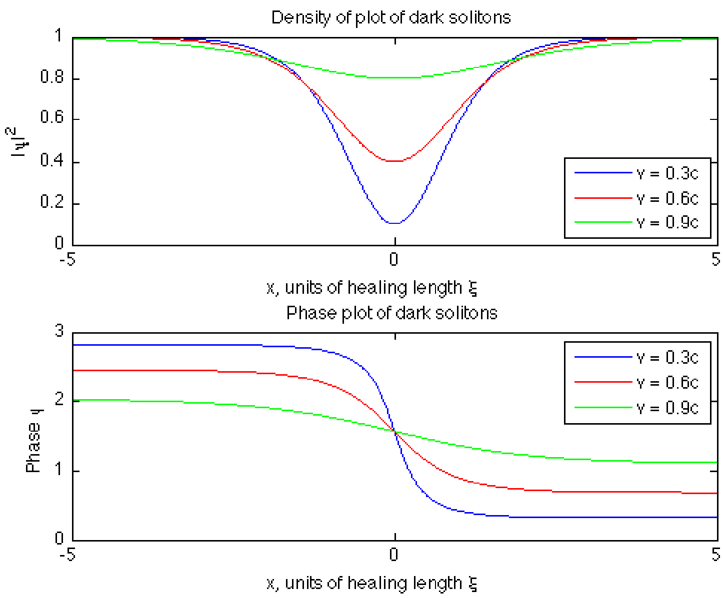

Figure 2.1: The density and phase of a dark soliton. The phase shift of a 'black' soliton, $v = 0$, is step shift of magnitude π. For increasing velocity the phase shift reduces and spreads spatially, disappearing in the limit $v \rightarrow c$.

Chapter 3

Mathematical Formalism of BEC

In this chapter we will examine the mathematical formalism used to describe a Bose-Einstein condensate. To begin with we look at the GPE mean field description. To gain a more complete description we separate the condensate and thermal background in an effective field theory. We look at the Wigner formalism and its ability to turn a field operator equation into an equivalent differential equation. We will find that with the right approximations we will be able to give a description of a Bose-Einstein condensate, which takes into account the effects of scattering and thermal noise, and that this description can be expressed as a stochastic differential equation. This will provide the basis of our Lagrangian variational method in chapter 4.

3.1 Gross-Pitaevskii Equation

The formation of a Bose-Einstein condensate require ultra-cool atoms. For high temperatures, the thermal energy is large enough that the atoms have a range of excited states available to them and the atoms are distributed in a range of states. For low temperatures a vast majority of the bosonic atoms are in the ground state and we have a Bose-Einstein condensate. There is a well defined phase transition which occurs at a temperature T_c. For $T < T_c$ the system will form a Bose-Einstein condensate. In practice to achieve such ultra-cool atoms, one uses laser cooling and then evaporative cooling to lower the thermal energy of the atoms $\epsilon < k_B T_c$. The atoms are isolated via a magnetic trap to prevent them gaining energy from their surroundings. In real experiments the trapping potential is usually harmonic, or can at least be modelled as harmonic under suitable conditions.

Bosons are integer spin particles, and hence the Pauli exclusion principle does not forbid multiple bosons from occupying the same quantum state. This has an important consequence at low temperatures. The thermodynamic properties of bosons differ from that of fermions, with the distribution of bosons at a given temperature obeying the Bose-Einstein distribution:

$$\bar{n}_{\mathrm{BE}}(\epsilon_\nu) = \frac{1}{e^{(\epsilon_\nu - \mu)/k_B T} - 1},$$

where $\bar{n}_{\mathrm{BE}}(\epsilon_\nu)$ is the mean occupation number of the single-particle state ν, ϵ_ν is the energy of the state ν; μ the chemical potential, k_B the Boltzmann constant and T the temperature. For low temperatures $T \sim \mu K$ we find that the low energy modes contain significant occupation and the high energy modes very little. At $T = 0$ it becomes favourable for all the bosons to occupy the lowest energy mode. In this case all bosons are in the same state and the state of the system is simply described by the position of the bosons. This allows us to write a simple 'global' wave function (called a classical field description) which describes the system and obeys a relatively simple dynamical formula, the Gross-Pitaevskii equation (GPE). In reality not all the bosons will be in the lowest mode, but the lowest few energy modes will have high occupation and there will be negligible occupation of the high energy states. The fact that any number of bosons occupy the same quantum state means that the different bosons can be described simply by their position. As we do not need to specify the state of each boson we can simply use a single wavefunction which describes the positions of each boson:

$$\psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, t) = \xi(\mathbf{x}_1, t)\xi(\mathbf{x}_2, t)\ldots\xi(\mathbf{x}_N). \tag{3.1}$$

The condensate can be described as a single wavefunction which is the product of the individual wavefunctions that describes the condensate. Therefore, a macroscopic sample containing tens of thousands of atoms may be described by a single wavefunction, without the usual complications associated with many body problems. The collection of atoms allows experimental verification of the probability amplitude, with the proportion of atoms in each state representing the probability of a single atom in each state. Analytically this single wavefunction allows for several exact solutions where analytic progress would normally be impossible.

3.1.1 GPE Derivation

To make these ideas more precise we look at the derivation of the GPE, gaining insight to the description and approximations made in the low temperature description of a BEC, as well as gaining familiarity with bosonic field operators. Ultimately we will find that the field theory description of the condensate can be approximated by a c-field function which satisfies a differential equation. To derive the GPE (following [18]), we start with the many-body Hamiltonian describing N interacting particles under the effects of an external potential V_{ext} and the particle-particle interaction potential $V(\mathbf{x}-\mathbf{x}')$ given in second quantization:

$$\begin{aligned}\hat{H} = &\int d\mathbf{x}^3 \hat{\Psi}^\dagger(\mathbf{x}) \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{x})\right] \hat{\Psi}(\mathbf{x}) \\ &+ \frac{1}{2}\int d\mathbf{x} d\mathbf{x}' \hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}^\dagger(\mathbf{x}') V(\mathbf{x}-\mathbf{x}')\hat{\Psi}(\mathbf{x})\hat{\Psi}(\mathbf{x}').\end{aligned}$$

$\hat{\Psi}^\dagger(\mathbf{x})$ and $\hat{\Psi}(\mathbf{x})$ are, respectively, the creation and annihilation boson field operators. They create and annihilate particles at position $\mathbf{x}$. For the vacuum state $|0\rangle$, we have $\hat{\Psi}^\dagger(\mathbf{x})|0\rangle$ producing a particle at position $\mathbf{x}$, while $\hat{\Psi}(\mathbf{x})|0\rangle = |0\rangle$. These states are in what is called Fock space. The boson field operator can be separated into the product of a wavefunction and annihilation operator as

$$\hat{\Psi}(\mathbf{x}) = \sum_i a_i \Psi_i(\mathbf{x}),$$

where the $\Psi_i(\mathbf{x})$ form an orthonormal set, a_i and $a_i^\dagger$ are the bosonic creation and annihilation operators satisfying

$$a_i^\dagger|n_0, n_1, \ldots, n_i, \ldots\rangle = \sqrt{n_i+1}|n_0, n_1, \ldots, n_i+1, \ldots\rangle$$
$$a_i|n_0, n_1, \ldots, n_i, \ldots\rangle = \sqrt{n_i}|n_0, n_1, \ldots, n_i-1, \ldots\rangle$$
$$[a_i^\dagger, a_{i'}] = \delta_{i,i'}, \quad [a_i, a_{i'}] = [a_i^\dagger, a_{i'}^\dagger] = 0.$$

If we move to the Heisenberg formulation, moving the time dependence to the boson field operators $\hat{\Psi}(\mathbf{x}, t)$, we can decompose them as follows

$$\hat{\Psi}(\mathbf{x}, t) = \langle\hat{\Psi}(\mathbf{x}, t)\rangle + \hat{\Psi}'(\mathbf{x}, t) = \Phi(\mathbf{x}, t) + \hat{\Psi}'(\mathbf{x}, t),$$

the sum of the order parameter or "wave function of the condensate" $\Phi(\mathbf{x}, t)$, and the perturbation $\hat{\Psi}'(\mathbf{x}, t)$ on that state. We note that the order parameter $\Phi(\mathbf{x}, t)$ is a function and not an operator. Next we use the Heisenberg equation of motion

$$i\hbar\frac{\partial\hat{\Psi}(\mathbf{x}, t)}{\partial t} = [\hat{\Psi}(\mathbf{x}, t), \hat{H}],$$

to give the time evolution of $\hat{\Psi}(\mathbf{x}, t)$. Substituting this in we find,

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}(\mathbf{x}, t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{x}) + \int d\mathbf{x}'\hat{\Psi}^\dagger(\mathbf{x}', t)V(\mathbf{x}'-\mathbf{x})\hat{\Psi}(\mathbf{x}', t)\right]\hat{\Psi}(\mathbf{x}, t).$$

So far everything has been an exact description in this second quantized regime. The Gross-Pitaevskii equation is a mean field approximation and describes the evolution of the condensate wave function $\Phi(\mathbf{x}, t)$, ignoring the effects of fluctuations. We now make three approximations in order to derive the GPE:

1. We are assuming $T \sim 0$, so that the fluctuations or perturbations from the order parameter are negligible i.e. $\hat{\Psi}(\mathbf{x}, t) \approx \Phi(\mathbf{x}, t)$.

2. The interaction potential is assumed to be the contact potential

   $$V(\mathbf{x}'-\mathbf{x}) = g\delta(\mathbf{x}'-\mathbf{x}) \quad g \equiv \frac{4\pi\hbar^2 a}{m},$$

   where a is the S-wave scattering length. The scattering length describes the scattering of two particles (equivalent to the scattering of a single particle in a potential) in the low energy limit.

3. The inter-particle spacing is much larger than a, making the details of the two-body potential unimportant.

Making these approximations gives us the famous Gross-Pitaevskii equation

$$i\hbar\frac{\partial\psi(\mathbf{x}, t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x}, t) + V(\mathbf{x})\psi(\mathbf{x}, t) + g|\psi(\mathbf{x}, t)|^2\psi(\mathbf{x}, t), \quad g = \frac{4\pi\hbar^2 a}{m}. \tag{3.2}$$

We have reduced a complex problem involving the description and interaction of many particles, to a simple wave equation that describes the average dynamics of the condensate via a nonlinear wave equation.

The time-dependent GPE is a mean-field equation, which describes the average dynamics of the condensate only. It contains no information of the condensate relaxation towards its equilibrium state, or fluctuations of the order parameter. Both these effects are included in the perturbation operator $\hat{\Psi}'(\mathbf{x}, t)$.

3.2 Quasi-one-dimensional systems

The Gross-Pitaevskii equation, while approximate, gives a good description of low temperature $T \ll T_c$ condensates and has a number of nice analytic properties. So before considering the possible corrections to the GPE for non-zero temperatures it can be illuminating to consider the techniques and approximations used to solve the Gross-Pitaevskii equation. In three dimensions the GPE is not an integrable system and hence we cannot find exact non-trivial solutions. However the one dimensional homogenous GPE ($V_{\text{ext}}(x) = 0$) is integrable and admits many exact solutions. This process of reducing the dimensionality of a problem to that of a (1+1) system, is rather common. While losing some possible phenomena, one dimensional systems often allow analytic progress to be made. Apart from being of great analytical tractability, one-dimensional solutions do have physical relevance. The one dimensional GPE can be used as an approximation to a cigar shaped potential aligned along the z-axis. Assuming the trapping potential is quadratic, as is usually the case, we say that a system is quasi-one-dimensional under the following description.

$$V_{\text{trap}}(\mathbf{x}) = \frac{m}{2}(\omega_\perp^2 r_\perp^2 + \omega_z^2 z^2),$$

where $\omega_z, \omega_\perp$ are the magnetic trap frequency in the z direction and the x, y directions respectively. The linear oscillator lengths, $a_\perp = \sqrt{\hbar/m\omega_\perp}$ and $a_z = \sqrt{\hbar/m\omega_z}$, define the length scales and provide the condition for which a one-dimensional approximation is valid. The condition is given as

$$\frac{a_\perp}{a_z} = \sqrt{\frac{\omega_z}{\omega_\perp}} \ll 1,$$

in [36] and

$$\frac{a_\perp}{\xi} \sim 2\sqrt{2\pi}\boldsymbol{\epsilon}, \quad \xi = (8\pi n a)^{-1/2},$$

where a is the s-wave scattering length, in [14]. This sort of potential constrains the atoms in the condensate in the axial direction. Essentially we require that the condensate is bound tightly in the transverse directions, such that it is free to only travel along one axis. In terms of energy we require that the lowest possible radial excitement, $\hbar\omega_\perp$, must exceed kinetic energy of the soliton so that the latter cannot be transferred into the BEC unstable transverse modes by interatomic interactions. In short we require $\mu \ll \hbar\omega_\perp$. In one dimension the interaction constant g is reduced to $g_{1\text{D}} = 2a\hbar\omega_\perp$.

3.2.1 Integrability of 1D GPE

The importance of the one-dimensional Gross-Pitaevskii equation, in terms of analytic solutions, is its integrability. The one-dimensional GPE, despite being nonlinear, is completely integrable [16, 35]. This means that in addition to having exact solutions there are an infinite number of conserved quantities; solvability through the inverse scattering method and N-soliton solutions. This is a much stronger condition than Painlevé integrability which is a criterion that only ensures exact solutions. There are lists of exact solotonic solutions of the one-dimensional GPE for a variety of special potentials, see for example [5, 32]. The solutions are usually found through the inverse transform and given one solution others can sometimes be found via the Darboux transformation and appropriate Lax pair. For example see [36] for a list of 27 different soliton solutions for different potentials. Exact solutions are invaluable in gaining a full understanding of a given system. The integrability of the GPE does not extend to higher dimensions.

The integrability properties of the one-dimensional GPE system allow us to make analytic progress. We will use exact dark soliton solutions of the one-dimensional GPE as an ansatz for treating the SGPE, which is not integrable. The idea is that although the SGPE cannot be described by an integrable differential equation it can be treated as a perturbation of the GPE, and as such we might expect that the exact solutions of the GPE would make approximate solutions if we allow the parameters to vary via collective variables, for details see Chapter 4. The soliton wavefunctions with collective variables to describe, width, depth and velocity do in fact provide a good approximation in a system described by the SGPE.

3.2.2 1D Non-linear Schrödinger equation

We can scale the one dimensional GPE to the following form (see section 4.4.1),

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \pm |u|^2 u = 0, \tag{3.3}$$

where the sign of the nonlinear term is determined by the sign of the scattering length. This equation is known as the Nonlinear Schrödinger equation (NLS). As the NLS equation is just a scaled version of the GPE, there exist soliton eigenstates, which can constructed via the inverse scattering transform. For details of the construction of dark soliton solutions via the inverse scattering transform see [48].

The existence of exact soliton solutions is certainly not unique to Bose-Einstein condensates, the GPE or NLS equations. In optics, the propagation of a light pulse in a nonlinear Kerr medium is also described by the NLSE in the appropriate scaling and solitons have been widely observed. Most of the early work in describing dark soliton solutions, in exactly integrable or nearly exactly integrable nonlinear wave solutions was done in the context of propagating dark solitons in optics. In fact almost any nonlinear wave equation admits soliton solutions. Furthermore these equations hold exact multiple soliton solutions and can describe interacting solitons.

The dynamics and interactions of a zero temperature or cold dark Bose-Einstein condensate is well known. The real interest recently is in the study of finite temperature BEC taking into account the thermal and quantum fluctuations, described by $\hat{\Psi}'(\mathbf{x}, t)$ in the GPE derivation.

3.3 Finite Temperature BEC

To give a more complete theoretical description of the behaviour of BEC's one must include both thermal and quantum fluctuations, as well as the dynamics of the system before it reaches equilibrium. At non-zero temperatures the distribution of energy among the atoms will mean the cooler atoms form part of a condensate while the hotter atoms constitute a thermal cloud. The condensate and the thermal cloud will undergo interactions leading to condensate growth or evaporation. The GPE does not account for fluctuations due to incoherent collisions between the condensate and non condensate atoms. Ultimately we will show that warm condensates can be described by a simple growth stochastic Gross-Pitaevskii equation. This equation is essentially the GPE with a dissipative term and stochastic noise. We aim to treat these extra terms as perturbations of the GPE in the application of the Lagrangian variational method.

3.4 C-field Formalism

As we have already seen a Bose-Einstein condensate can be described by a classical field equation at zero temperature, via the Gross-Pitaevskii equation. Blakie *et al.* [9] use the term *c-field* descriptions, for a technique which describes a quantum system in terms of some classical field. The requirement for this description to be accurate is that many modes of the system are highly occupied. It is these highly Bose degenerate states that are described by a classical field. The main purpose of this chapter is to show that a more complete description of the dynamics of a BEC can be described in the c-field formalism in the form of a modified GPE. We outline the formulation of the stochastic Gross-Pitaevskii equation (SGPE).

3.5 Effective Field theory

As in the case of the GPE, we begin the description of the quantum system with the second quantized Hamiltonian for N interacting Bosons

$$\begin{aligned}\hat{H}(t) = &\int d^3\mathbf{x}\hat{\Psi}^\dagger(\mathbf{x},t)H_{\mathrm{sp}}\hat{\Psi}(\mathbf{x},t)\\ &+\frac{1}{2}\int d^3\mathbf{x}\int d^3\mathbf{x}'\hat{\Psi}^\dagger(\mathbf{x},t)\Psi^\dagger(\hat{\mathbf{x}'},t)V(\mathbf{x}-\mathbf{x}')\Psi(\hat{\mathbf{x}'},t)\Psi(\hat{\mathbf{x}},t),\end{aligned}$$

where the single particle Hamiltonian is

$$H_{\mathrm{sp}} = -\frac{\hbar^2\nabla^2}{2m} + V_{\mathrm{ext}}(\mathbf{x}),$$

and $\hat{\Psi}(\mathbf{x},t)$ is once again the Bosonic field operator which annihilates a particle at position $\mathbf{x}$ at time t. Now we choose a subspace of the full Hilbert space on which we will consider a full quantum treatment by ignoring the high energy states $\epsilon > E_{\mathrm{max}}$. This eliminates states with momentum $\hbar\Lambda(\mathbf{x}) \approx \sqrt{2m(E_{\mathrm{max}} - V_{\mathrm{ext}}(\mathbf{x}))}$ and 'coarse grains' our description to length scales $1/\Lambda(\mathbf{x})$. For a typical Bose-Einstein condensate the temperature is of order 1μK, so that the de Broglie wavelength of the atoms is much larger than the inter-particle spacing. The lengths of interest are larger than the interaction range and the atomic level details are unimportant. We can eliminate the short wavelength modes by removing the modes corresponding to energies greater than some E_{max}. In essence we are considering the system projected onto a low energy subspace $\mathbf{L}$. We define $\mathbf{L}$ to be $\mathbf{L} = \{n|\epsilon_n < E_{\mathrm{max}}\}$. This cutoff allows us to treat the interaction potential as a delta function on our long length scale, so that the scattering between two atoms is parameterized by the S-wave scattering length as in the GPE derivation. This description of atomic interactions is an *effective field theory*, and the effective Hamiltonian becomes

$$\hat{H}_{\text{eff}} = \int d^3\mathbf{x}\hat{\psi}^\dagger(\mathbf{x})H_{\text{sp}}\hat{\psi}(\mathbf{x}) + \frac{g}{2}\int d^3\mathbf{x}\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{x}), \tag{3.4}$$

where

$$g = \frac{4\pi\hbar^2 a}{m},$$

as before. The difference here from the GPE derivation is that we are using *coarse-grained* field operators $\hat{\psi}(\mathbf{x})$, essentially the projection of the bosonic field operators on the eigenspace $\mathbf{L}$.

$$\hat{\psi}(\mathbf{x}) = \sum_{n\in\mathbf{L}} \hat{a}_n\phi_n(\mathbf{x}),$$

where $\phi_n(\mathbf{x})$ are the single particle eigenstates of H_{sp} with energy ϵ_n and $\hat{a}_n$ are the destruction operators. In this case the commutation relations of the $\hat{\psi}(\mathbf{x})$ are no longer delta functions as the were for the bosonic field operators but become a course-grained delta function

$$[\hat{\psi}(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{x})] = \delta_{\mathbf{L}}(\mathbf{x} - \mathbf{x}'),$$

which acts as a delta function on $\mathbf{L}$ and doubles as the projector operator kernel.

Just as for the GPE we can use the Heisenberg equation of motion and the commutation relations to give the equation of motion of the coarse grain field operator within the subspace $\mathbf{L}$,

$$i\hbar\frac{\partial\hat{\psi}(\mathbf{x})}{\partial t} = \int d^3\mathbf{x}'\delta_{\mathbf{L}}(\mathbf{x} - \mathbf{x}')\left\{H_{\text{sp}}\hat{\psi}(\mathbf{x}') + g\hat{\psi}^\dagger(\mathbf{x}')\hat{\psi}(\mathbf{x}')\hat{\psi}(\mathbf{x}')\right\}. \tag{3.5}$$

This equation governs the dynamics of the low energy field operator. Unfortunately this is an operator equation over a restricted subspace, which means it is not in practice very different to solving the full field theory. We can use c-field techniques to create suitable approximations and yield a more tractable dynamical equation.

3.6 Projection onto the Classical Field Region

In creating our effective field theory we have restricted ourselves to the low energy subspace $\mathbf{L}$. As it turns out this subspace is much too large to be of any use computationally. To simplify matters we consider splitting the system into two sections (see figure 3.1). A region of significant Bose degeneracy which we will describe with a c-field and a weakly occupied region which is assumed to be thermalized. We describe them as follows:

1. The *condensate band* (**C**) consisting of low energy modes with significant occupation. This is the region we wish to describe using the classical field describing the condensate and low energy excitations for temperatures $T < T_c$.

2. The *non-condensate band* (**NC**) contains all modes in $\mathbf{L}$ that are not in $\mathbf{C}$. These are weakly occupied modes that are essentially thermalized.

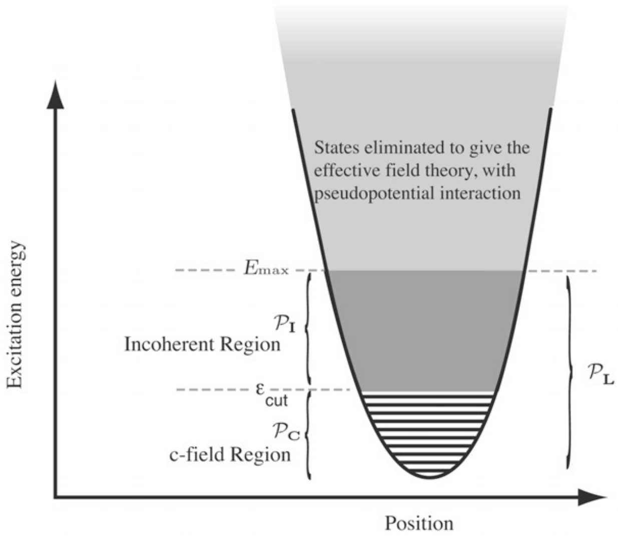

Figure 3.1: A schematic showing the condensate band, non-condensate band, and the states eliminated to give the effective field theory. In the Stochastic Gross-Pitaevskii equation, the condensate band is treated quantum mechanically in a classical field approximation. The non-condensate band is treated semi-classically, and is described as a grand canonical reservoir.

We differentiate these two bands by a single particle energy ϵ_{cut}, where $\mathbf{C}$ has energies $\epsilon \leq \epsilon_{\text{cut}}$ and $\mathbf{NC}$ corresponds to $\epsilon_{\text{cut}} < \epsilon < E_{\max}$. To restrict the field operator to these regions we use orthogonal projection operators, defined on an arbitrary function $F(\mathbf{x})$ by

$$\mathcal{P}_{\mathbf{C}}\left\{F(\mathbf{x})\right\} \equiv \sum_{n\in\mathbf{C}} \phi_n(\mathbf{x}) \int d^3\mathbf{x}' \phi^*(\mathbf{x}') F(\mathbf{x}'),$$

$$\mathcal{P}_{\mathbf{NC}}\left\{F(\mathbf{x})\right\} \equiv \sum_{n\in\mathbf{NC}} \phi_n(\mathbf{x}) \int d^3\mathbf{x}' \phi^*(\mathbf{x}') F(\mathbf{x}'),$$

with $\mathbf{C} = \{n : \epsilon_n \leq \epsilon_{\text{cut}}\}$ and $\mathbf{NC} = \{n : \epsilon_{\text{cut}} < \epsilon_n \leq E_{\max}\}$. It is clear that

$$\mathbf{L}{=}\mathbf{C}{+}\mathbf{NC}, \quad \mathcal{P}_{\mathbf{C}}\mathcal{P}_{\mathbf{NC}} = 0, \quad [\mathcal{P}_{\mathbf{C}}, \mathcal{P}_{\mathbf{NC}}] = 0 \;\text{ and }\; \mathcal{P}^2_{\mathbf{C/NC}} = \mathcal{P}_{\mathbf{C/NC}}$$

as with all orthogonal linear projections.

The field operator can be projected onto the two regions

$$\hat{\psi}_{\mathbf{C}}(\mathbf{x}) \equiv \mathcal{P}_{\mathbf{C}}\left\{\hat{\psi}(\mathbf{x})\right\} = \sum_{n\in\mathbf{C}} \hat{a}_n\phi_n(\mathbf{x}),$$

$$\hat{\psi}_{\mathbf{NC}}(\mathbf{x}) \equiv \mathcal{P}_{\mathbf{NC}}\left\{\hat{\psi}(\mathbf{x})\right\} = \sum_{n\in\mathbf{NC}} \hat{a}_n\phi_n(\mathbf{x}).$$

The commutator relations for the two band operators are not Dirac functions but

$$[\hat{\psi}_{\mathbf{C}}(\mathbf{x}), \hat{\psi}^\dagger_{\mathbf{C}}(\mathbf{x})] = \delta_{\mathbf{C}}(\mathbf{x}, \mathbf{x}'),$$

where

$$\delta_{\mathbf{C}}(\mathbf{x}, \mathbf{x}') = \sum_{n\in\mathbf{C}} \phi_n(\mathbf{x})\phi^*_n(\mathbf{x})$$

is the kernel of the projector $\mathcal{P}_{\mathbf{C}}$. Although not a true delta function $\delta_{\mathbf{C}}(\mathbf{x}, \mathbf{x}')$ behaves like one in the condensate band.

3.6.1 Hamiltonian decomposition

Using the band decomposition we find that the effective Hamiltonian (3.4) can be decomposed in terms of $\hat{\psi}_{\mathbf{C}}(\mathbf{x})$ and $\hat{\psi}_{\mathbf{NC}}(\mathbf{x})$ to give

$$\hat{H}_{\text{eff}} = \hat{H}_C = \hat{H}_C + \hat{H}_{NC} + \hat{H}_{I,C}$$

where the term $\hat{H}_C$ involves only the condensate band field operators, $\hat{H}_{NC}$ only the non-condensate band field operators and

$$\hat{H}_{I,C} = \hat{H}^{(1)}_{I,C} + \hat{H}^{(2)}_{I,C} + \hat{H}^{(3)}_{I,C}, \tag{3.6}$$

represents the atomic interaction of terms involving one, two or three condensate band operators. These are given explicitly as

$$\hat{H}_C = \int d^3\mathbf{x}\hat{\psi}^{\dagger}_{\mathbf{C}}(\mathbf{x})H_{\text{sp}}\hat{\psi}_{\mathbf{C}}(\mathbf{x}) + \frac{g}{2}\int d^3\mathbf{x}\hat{\psi}^{\dagger}_{\mathbf{C}}(\mathbf{x})\hat{\psi}^{\dagger}_{\mathbf{C}}(\mathbf{x})\hat{\psi}_{\mathbf{C}}(\mathbf{x})\hat{\psi}_{\mathbf{C}}(\mathbf{x}) \tag{3.7}$$

$$\hat{H}_{NC} = \int d^3\mathbf{x}\hat{\psi}^{\dagger}_{\mathbf{NC}}(\mathbf{x})H_{\text{sp}}\hat{\psi}_{\mathbf{NC}}(\mathbf{x}) + \frac{g}{2}\int d^3\mathbf{x}\hat{\psi}^{\dagger}_{\mathbf{NC}}(\mathbf{x})\hat{\psi}^{\dagger}_{\mathbf{NC}}(\mathbf{x})\hat{\psi}_{\mathbf{NC}}(\mathbf{x})\hat{\psi}_{\mathbf{NC}}(\mathbf{x}) \tag{3.8}$$

$$\hat{H}^{(1)}_{I,C} = g\int d^3\mathbf{x}\hat{\psi}^{\dagger}_{\mathbf{NC}}(\mathbf{x})\hat{\psi}^{\dagger}_{\mathbf{NC}}(\mathbf{x})\hat{\psi}_{\mathbf{C}}(\mathbf{x})\hat{\psi}_{\mathbf{C}}(\mathbf{x}) + \mathbf{h.c.} \tag{3.9}$$

$$\begin{aligned}\hat{H}^{(2)}_{I,C} &= g\int d^3\mathbf{x}\hat{\psi}^{\dagger}_{\mathbf{NC}}(\mathbf{x})\hat{\psi}^{\dagger}_{\mathbf{NC}}(\mathbf{x})\hat{\psi}_{\mathbf{C}}(\mathbf{x})\hat{\psi}_{\mathbf{C}}(\mathbf{x}) + \mathbf{h.c.} \\ &\quad + \frac{g}{2}\int d^3\mathbf{x}\hat{\psi}^{\dagger}_{\mathbf{NC}}(\mathbf{x})\hat{\psi}^{\dagger}_{\mathbf{NC}}(\mathbf{x})\hat{\psi}_{\mathbf{NC}}(\mathbf{x})\hat{\psi}_{\mathbf{NC}}(\mathbf{x}) + \mathbf{h.c.}\end{aligned} \tag{3.10}$$

$$\hat{H}^{(3)}_{I,C} = g\int d^3\mathbf{x}\hat{\psi}^{\dagger}_{\mathbf{C}}(\mathbf{x})\hat{\psi}^{\dagger}_{\mathbf{C}}(\mathbf{x})\hat{\psi}_{\mathbf{C}}(\mathbf{x})\hat{\psi}_{\mathbf{NC}}(\mathbf{x}) + \mathbf{h.c.} \tag{3.11}$$

where **h.c.** is the Hermitian conjugate. The terms involving one $\hat{\psi}_{\mathbf{NC}}(\mathbf{x})$ and one $\hat{\psi}_{\mathbf{C}}(\mathbf{x})$ do not provide any contribution as the non-condensate band operators have no mean field.

3.6.2 C-Field Regimes

In order to make further process we need to make approximations to deal with the interaction Hamiltonians containing field operators of both the condensate and non-condensate band. The three regimes, each valid under different conditions, are:

1. **Projected Gross-Pitaevskii Equation (PGPE)**
   The PGPE is simply the Gross-Pitaevskii equation where the evolution is restricted to the condensate band. This formalism is a micro-canonical approach where the condensate band is isolated from the non-condensate band, there is no exchange of energy or particles. The two bands are uncoupled and all the cross terms in the interaction Hamiltonian are dropped. This is a good description at $T \approx 0$ when essentially all atoms are in the condensate band.

2. **Truncated Wigner Projected Gross-Pitaevskii Equation (TWPGPE)**
   The TWGPE is appropriate when there are modes of low occupation in the condensate band. In this case the population of the non-condensate band is negligible and once again the cross terms in the interaction Hamiltonian can be dropped. This formulation differs from that of the PGPE through the addition of stochastic noise. The modes of low occupation should include quantum fluctuations if they are to accurately describe the system quantum mechanically. We can approximate the quantum fluctuations through sampling of the initial Wigner distribution, allowing us to describe the equilibrium dynamics of a BEC at low temperatures $T \ll T_c$. While we are primarily interested in high temperature condensates the Wigner formalism is central to the SPGPE.

3. **Stochastic Projected Gross-Pitaevskii Equation (SPGPE)**
   In this formalism we retain all the terms describing interactions between the **C** and **NC** regions, and hence the SGPE is the most complete description. The inclusion of the interaction terms means that we are treating the condensate as a grand canonical system which may exchange both energy and particles with the non-condensate band. This exchange happens via scattering processes. Using the Wigner formalism we will show that the inclusion of scattering is equivalent to finding a stochastic differential equation with dissipative and stochastic noise terms. This more complete description is appropriate at high temperature where there is a significant thermal component.

The SPGPE provides the most complete description of a Bose-Einstein condensate, especially at high temperatures. The fact that the SPGPE is equivalent to a dissipative stochastic differential equation is of importance as it means we will be able treat the SPGPE as a perturbed GPE and use the Lagrangian variational method to solve this perturbed equation. It is enlightening to look at the Wigner Formalism to see how this correspondence between the operator evolution equation (3.5) is transformed into a Fokker-Planck equation in phase space and finally to a stochastic differential equation.

3.7 Wigner Formalism

3.7.1 Overview

By Describing our system using c-field methods, introducing an energy cutoff ϵ_{cut}, we allow the Bose-degenerate system to be described by a classical field. We can map the quantum density operator ρ to a quasi-probability Wigner function. The quasi probability can be shown to be equivalent to a Fokker-Planck equation with a drift term. The Fokker-Planck equation is in turn equivalent to a stochastic differential equation for the classical field. This stochastic differential equation is much easier to solve than the operator equations in Hilbert space described in 3.6.1. The Wigner formalism is important as it is the key mapping connecting the projected Hamiltonian equations and the modified SGPE. The Wigner function and the phase space representation of the quantum field is a well established topic and a detailed discussion can be found in Gardiner's and Zoller's book [28].

3.7.2 Quasi-probability distribution and Coherent States

The Wigner function is a quasi-probability distribution, essentially this means that the Wigner function is almost a probability distribution except it takes on negative values. The conditions on the Wigner function imply that for most states it is negative somewhere, with the exception of the coherent states (linear combinations of coherent states), the squeezed coherent states and Gaussian density operators.

Quantum mechanics, unlike classical mechanics does not allow us to describe a state as a single point in phase space and describe its evolution as a path in phase space. To find the closest classical type behaviour we have to look at the minimal uncertainty states. The Wigner function is positive for such states. The coherent states are states associated with the quantum harmonic oscillator, which display maximal coherence and classical like behaviour, coherent states evolve in time (under the harmonic oscillator Hamiltonian) through phase space without spreading or distorting. The coherent states are minimum uncertainty states satisfying

$$a|\alpha\rangle = \alpha|\alpha\rangle,$$

where a is the annihilation operator and α a complex number. The coherent state is left unchanged by the detection (or annihilation of a particle). These properties of the coherent states make them a natural basis in which to represent the Wigner function.

3.7.3 Wigner Representation of a Single Quantum Mode

If a system has a state vector $|\psi, t\rangle$, then the density operator is defined by the outer product

$$\rho(t) = |\psi, t\rangle\langle\psi, t|.$$

Phase space methods rely on expressing the density operator in terms of a c-number function of a coherent state variable α in phase space. The Wigner representation makes the connection through the Wigner function, which expresses the density operator in a basis of coherent states. The Wigner function $W(\alpha, \alpha^*)$ was introduced by Wigner in 1932 [46], and is defined as the Fourier transform of the symmetrically ordered quantum characteristic function

$$W(\alpha, \alpha^*) = \frac{1}{\pi^2}\int d^2\lambda \exp\left[-\lambda\alpha^* + \lambda^*\alpha\right]\chi(\lambda, \lambda^*),$$

where the symmetrically ordered characteristic function is given by

$$\chi(\lambda, \lambda^*) = \mathrm{tr}\left\{\rho \exp\left[\lambda a^\dagger - \lambda^* a\right]\right\}.$$

The Wigner function exists for any density operator [28], so for any density operator we can always find this c-number function $W(\alpha, \alpha^*)$. The usefulness of the Wigner function and the phase space representation is the fact that the creation and destruction operators acting on any given density ρ are mapped to differential operators on the corresponding

Wigner function as follows [28]:

$$a\rho \longleftrightarrow \left(\alpha + \frac{1}{2}\frac{\partial}{\partial\alpha^*}\right) W(\alpha,\alpha^*) \tag{3.12}$$

$$a^\dagger\rho \longleftrightarrow \left(\alpha^* + \frac{1}{2}\frac{\partial}{\partial\alpha}\right) W(\alpha,\alpha^*) \tag{3.13}$$

$$\rho a \longleftrightarrow \left(\alpha - \frac{1}{2}\frac{\partial}{\partial\alpha^*}\right) W(\alpha,\alpha^*) \tag{3.14}$$

$$\rho a^\dagger \longleftrightarrow \left(\alpha^* + \frac{1}{2}\frac{\partial}{\partial\alpha}\right) W(\alpha,\alpha^*). \tag{3.15}$$

It is these operator mappings which give us a differential equation in terms of $W(\alpha, \alpha^*)$, from which we can identify a Fokker-Planck equation and ultimately produce a stochastic differential equation.

3.7.4 Wigner Representation of the Quantum Field

Having described a single mode in terms of the Wigner function, we can quite naturally extend this description to the full multimode system. The generalization for a system with M modes is given in [9]

$$W_{\mathbf{C}}(\boldsymbol{\alpha},\boldsymbol{\alpha}^*) = \frac{1}{\pi^{2M}}\int d^2\boldsymbol{\lambda}\exp\left(\boldsymbol{\lambda}^\dagger\boldsymbol{\alpha} - \boldsymbol{\alpha}^\dagger\boldsymbol{\lambda}\right)\chi\left(\boldsymbol{\lambda},\boldsymbol{\lambda}^*\right), \tag{3.16}$$

with $\boldsymbol{\alpha} = [\alpha_0, \alpha_1, \ldots, \alpha_{M-1}]$. $\chi(\boldsymbol{\lambda}, \boldsymbol{\lambda}^*)$ is the symmetric quantum characteristic function for the condensate band density operator $\rho_{\mathbf{C}}$, where

$$\int d^2\boldsymbol{\alpha} \equiv \prod_{n\in\mathbf{C}}\int d^2\alpha_n.$$

The mode amplitudes are the relative weightings of the single particle energy eigenstates, so that the classical field $\psi_{\mathbf{C}}$ is given by the linear combination

$$\psi_{\mathbf{C}}(\mathbf{x}) = \sum_{n\in\mathbf{C}}\alpha_n\phi_n(\mathbf{x}).$$

We can see that the Wigner distribution predicts quantum noise in the form of vacuum noise. The field density average is given by

$$\begin{aligned}\int d^2\boldsymbol{\alpha}|\psi_{\mathbf{C}}(\mathbf{x})|^2 W_{\mathbf{C}}(\boldsymbol{\alpha},\boldsymbol{\alpha}^*) &= \left\langle\frac{\hat{\Psi}^\dagger_{\mathbf{C}}(\mathbf{x})\hat{\Psi}_{\mathbf{C}}(\mathbf{x}) + \hat{\Psi}_{\mathbf{C}}(\mathbf{x})\hat{\Psi}^\dagger_{\mathbf{C}}(\mathbf{x})}{2}\right\rangle, \\ &= \left\langle\hat{\Psi}^\dagger_{\mathbf{C}}(\mathbf{x})\hat{\Psi}_{\mathbf{C}}(\mathbf{x})\right\rangle + \frac{\delta_{\mathbf{C}}(\mathbf{x},\mathbf{x})}{2}.\end{aligned}$$

This shows that for each mode of noise there is half a quantum of vacuum noise in the commutator $\delta_{\mathbf{C}}(\mathbf{x}, \mathbf{x})$. We simulate stochastic noise in the TWPGPE regime by sampling the Wigner function, so it is comforting to see that the Wigner function does actually predict noise.

We have multimode versions of the correspondence between creation and annihilation operators on the density operator and differential operators on the Wigner function. To make this clearer we define the *projected functional derivative operators*

$$\frac{\bar{\delta}}{\bar{\delta}\psi_{\mathbf{C}}(\mathbf{x})} \equiv \sum_{n\in\mathbf{C}} \phi_n^*(\mathbf{x})\frac{\partial}{\partial\alpha_n},$$

$$\frac{\bar{\delta}}{\bar{\delta}\psi_{\mathbf{C}}^*(\mathbf{x})} \equiv \sum_{n\in\mathbf{C}} \phi_n(\mathbf{x})\frac{\partial}{\partial\alpha_n^*}.$$

In terms of these projected functional derivatives we have the following relationships

$$\hat{\psi}_{\mathbf{C}}(\mathbf{x})\rho_{\mathbf{C}} \longleftrightarrow \left(\psi_{\mathbf{C}}(\mathbf{x}) + \frac{1}{2}\frac{\bar{\delta}}{\bar{\delta}\psi_{\mathbf{C}}^*(\mathbf{x})}\right)W_{\mathbf{C}}, \tag{3.17}$$

$$\hat{\psi}_{\mathbf{C}}^\dagger(\mathbf{x})\rho_{\mathbf{C}} \longleftrightarrow \left(\psi_{\mathbf{C}}^*(\mathbf{x}) - \frac{1}{2}\frac{\bar{\delta}}{\bar{\delta}\psi_{\mathbf{C}}(\mathbf{x})}\right)W_{\mathbf{C}}, \tag{3.18}$$

$$\rho_{\mathbf{C}}\hat{\psi}_{\mathbf{C}}^\dagger(\mathbf{x}) \longleftrightarrow \left(\psi_{\mathbf{C}}(\mathbf{x}) - \frac{1}{2}\frac{\bar{\delta}}{\bar{\delta}\psi_{\mathbf{C}}^*(\mathbf{x})}\right)W_{\mathbf{C}}, \tag{3.19}$$

$$\rho_{\mathbf{C}}\hat{\psi}_{\mathbf{C}}^\dagger(\mathbf{x}) \longleftrightarrow \left(\psi_{\mathbf{C}}^*(\mathbf{x}) + \frac{1}{2}\frac{\bar{\delta}}{\bar{\delta}\psi_{\mathbf{C}}(\mathbf{x})}\right)W_{\mathbf{C}}. \tag{3.20}$$

These are the operator correspondences which turn the equations of motion of $\rho_{\mathbf{C}}$ to an equivalent differential equation for the equation of motion of the Wigner function for a multimode system.

3.7.5 The Truncated Wigner Approximation

The evolution of the density operator in the condensate band is given by von Neumann's equation

$$i\hbar\frac{\partial\hat{\rho}_{\mathbf{C}}(t)}{\partial t} = \left[\hat{H}_{\mathbf{C}}, \hat{\rho}_{\mathbf{C}}(t)\right],$$

where we neglect all interactions between the condensate and non-condensate band as we are working in the truncated Wigner regime. We recast this equation into a more manageable form via the mapping to the equation of motion for the Wigner function. Using the results (3.17)-(3.20) we find

$$\begin{aligned}\frac{\partial W_{\mathbf{C}}}{\partial t}\,\Big|_{\hat{H}_{\mathbf{C}}} = \int d^3\mathbf{x}\Bigg\{&\frac{ig}{4\hbar}\frac{\bar{\delta}}{\bar{\delta}\psi_{\mathbf{C}}(\mathbf{x})\bar{\delta}\psi_{\mathbf{C}}^*(\mathbf{x})}\psi_{\mathbf{C}}^*(\mathbf{x})\frac{\bar{\delta}}{\bar{\delta}\psi_{\mathbf{C}}^*(\mathbf{x})\bar{\delta}} + \mathbf{h.c} \\ &+ \frac{i}{\hbar}\frac{\bar{\delta}}{\bar{\delta}\psi_{\mathbf{C}}(\mathbf{x})}(H_{\mathrm{sp}} + g[|\psi_{\mathbf{C}}(\mathbf{x})|^2 - \delta_{\mathbf{C}}(\mathbf{x},\mathbf{x})])\psi_{\mathbf{C}}(\mathbf{x}) + \mathbf{h.c}\Bigg\}W_{\mathbf{C}}.\end{aligned} \tag{3.21}$$

The equation is rather intractable due to the third order derivatives. If we neglect these third order terms then we find that we have a Fokker-Planck equation with a a drift term but no diffusion term,

$$\frac{\partial W_{\mathbf{C}}}{\partial t}\,\Big|_{\hat{H}_{\mathbf{C}}} = \int d^3\mathbf{x}\left\{\frac{i}{\hbar}\frac{\bar{\delta}}{\bar{\delta}\psi_{\mathbf{C}}(\mathbf{x})}(H_{\mathrm{sp}} + g[|\psi_{\mathbf{C}}(\mathbf{x})|^2 - \delta_{\mathbf{C}}(\mathbf{x},\mathbf{x})])\psi_{\mathbf{C}}(\mathbf{x}) + \mathbf{h.c}\right\}W_{\mathbf{C}}. \tag{3.22}$$

The approximation made by neglecting the third order derivatives is the *truncated Wigner approximation*. In addition to being rather convenient mathematically this approximation is actually valid in many circumstances [9]. Having negligible third derivative terms is equivalent to the requirement that the modes in the c-field have high occupation.

The property of Fokker-Planck equations that is of importance to us is the fact Fokker-Planck equations can be mapped to an equivalent stochastic partial differential equation if the diffusion matrix is positive semidefinite [24]. There is a stochastic equation corresponding to the truncated Wigner approximation; but only when the third order terms are neglected. Finding the corresponding stochastic differential equation, we get the truncated Wigner projected Gross-Pitaevskii equation (TWPGPE),

$$i\hbar\frac{\partial\psi_{\mathbf{C}}(\mathbf{x})}{\partial t} = \mathcal{P}_{\mathbf{C}}\left\{\left(H_{\mathrm{sp}} + g\left[|\psi_{\mathbf{C}}(\mathbf{x})|^2 - \delta_{\mathbf{C}}(\mathbf{x},\mathbf{x})\right]\right)\psi_{\mathbf{C}}(\mathbf{x})\right\}. \tag{3.23}$$

The stochastic nature of the TWGPE is not immediately apparent, as it does not include some explicit randomly fluctuating function. The stochastic fluctuations come from the sampling of the initial state according to its Wigner function, which introduces a randomness. For a vacuum the randomness is of the form of occupation of half a quantum per mode as shown in section 3.7.4, other states have randomness which is associated with the particular quantum state. While the TWGPE is not used to describe high temperature condensates the method by which we converted a Heisenberg equation into a stochastic equation is essential to the development of the SPGPE.

3.8 The Projected Gross-Pitaevskii Equation

The PGPE is obtained by applying the projector operator $\mathcal{P}_{\mathbf{C}}$ to the Heisenberg equation of motion (3.5). The derivation follows the derivation of the GPE in section 3.1.1, except the condensate is reduced to the subspace $\mathbf{C}$, and the modes outside are uncoupled to the condensate and essentially ignored. For details see [19].

$$i\hbar\frac{\partial\psi_{\mathbf{C}}(\mathbf{x})}{\partial t} = \mathcal{P}_{\mathbf{C}}\left\{\left(H_{\mathrm{sp}} + g|\psi_{\mathbf{C}}(\mathbf{x})|^2\right)\psi_{\mathbf{C}}(\mathbf{x})\right\}. \tag{3.24}$$

The projected GPE is actually quite powerful. The PGPE, while uncoupled to the non-condensate band will evolve general configurations of the coherent region $\mathbf{C}$ towards an equilibrium. Although the equation is unitary and reversible we still expect that it will evolve general states into thermal equilibrium. This is because deterministic nonlinear systems display chaotic and hence ergodic behaviour if there are many degrees of freedom.

3.9 The Stochastic Gross-Pitaevskii Equation

The SPGPE is the end goal for us, the description of a warm Bose-Einstein condensate. This is a grand canonical approach in which the condensate is free to exchange energy and particles with the non-condensate thermal band. The thermal band is characterized by the temperature T and the chemical potential μ. The interaction terms are not neglected in the von Neumann equation describing the evolution of the density operator (3.6). The interactions are included in form of a dissipative and additive stochastic noise term adjoined to the PGPE. Ultimately we aim to treat these dissipative and stochastic noise terms as perturbations to the GPE. The full derivation of the master equation arising from the effective Hamiltonian and the subsequent formulation of the stochastic differential equations leading to the SPGPE is well known [11, 25]. The calculations are rather involved and we will not repeat them here but instead give an outline of the results.

3.9.1 The Master equation

The equation of motion of the density operator for the full system including both the condensate and non-condensate bands is given by the von Neumann equation

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}\left[\hat{H}_C + \hat{H}_{N,C} + \hat{H}_{I,C}, \rho\right]. \tag{3.25}$$

The standard procedure for using phase-space method is based on finding a master equation for the reduced system. We describe the condensate density operator, by eliminating the reservoir degrees of freedom [26–28, 31]. This is done by defining the condensate band density operator as the trace of ρ over the non-condensate band degrees of freedom

$$\rho_C \equiv \mathrm{tr}_{NC}\left\{\rho\right\}.$$

The full effective Hamiltonian is used to retain all the interaction terms. There are a number of different terms arising based on the interactions and can be categorized for the reduced system as follows

$$\dot{\rho}_C = \dot{\rho}_C|_{\mathrm{Ham}} \tag{3.26a}$$
$$+ \dot{\rho}_C|_{\mathrm{growth}} \tag{3.26b}$$
$$+ \dot{\rho}_C|_{\mathrm{scatt}}. \tag{3.26c}$$

We can see there are three separate components of the dynamics of the density operator of the condensate. The first term (3.26a) is the term from von Neumann's equation involving only the condensate band field operator. This is the only term used in the derivation of the TWPGPE. The other terms describe the effects of growth through exchange of particles with the non-condensate band (3.26b) and exchange of energy through scattering (3.26c).

3.9.2 The Stochastic Gross-Pitaevskii Equation

Just as for the case of the TWPGPE we map the density evolution equation, in this case the full master equation (3.26), into a differential equation of motion for the Wigner function. Again the truncated Wigner approximation is made and the third order derivatives are dropped to give a Fokker Planck equation which does contain diffusion terms. Once again this satisfies the requirement to be mapped to the associated stochastic differential

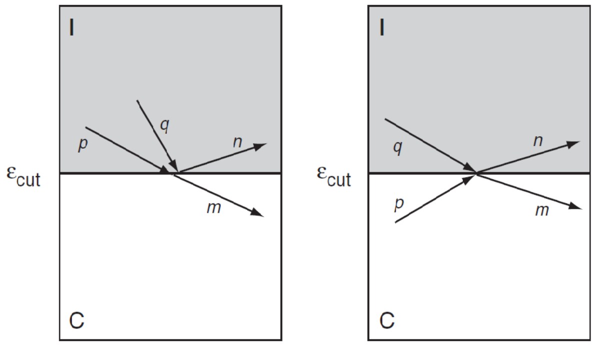

Figure 3.2: Growth and scattering process caused by interactions between the condensate and non-condensate bands. The growth process (a) occurs when two non-condensate band atoms collide, energy is transferred to one of the atoms, while the other enters the condensate band. The scattering process (b) occurs when a condensate and non-condensate atom collide in a number conserving process, leaving the populations of the bands unchanged.

equation. We eventually get the full SPGPE

$$(S)\mathrm{d}\psi_{\mathbf{C}}(\mathbf{x},t) = \mathcal{P}_{\mathbf{C}}\Bigg\{-\frac{i}{\hbar}\left(H_{\text{sp}} + g|\psi_{\mathbf{C}}(\mathbf{x})|^2\right)\psi_{\mathbf{C}}(\mathbf{x})\mathrm{d}t \tag{3.27a}$$

$$+\frac{G(\mathbf{x})}{k_BT}\left[\mu - \left(H_{\text{sp}} + g|\psi_{\mathbf{C}}(\mathbf{x}))|^2\right)\right]\psi_{\mathbf{C}}(\mathbf{x})\mathrm{d}t + dW_G(\mathbf{x},t) \tag{3.27b}$$

$$+\int d^3\mathbf{x}' M(\mathbf{x}-\mathbf{x}')\frac{i\hbar\nabla\cdot\mathbf{j}_{\mathbf{C}}(\mathbf{x}')}{k_BT}\psi_{\mathbf{C}}(\mathbf{x})\mathrm{d}t + i\psi_{\mathbf{C}}(\mathbf{x})dW_M(\mathbf{x},t)\Bigg\} \tag{3.27c}$$

where the (S) is placed to explicitly remind us that the stochastic differential equation is of the Sratonovich form.

The first term of the SPGPE (3.27a) is simply the PGPE and accounts for the evolution without any coupling between the condensate and non-condensate band. The effects of growth through the exchange of particles is described by the second term (3.27b), and the effects of energy via number conserving scattering is described by the third term (3.27c). As promised the effects of finite temperature are described by a GPE type equation with dissipation and stochastic noise terms.

3.9.3 Growth and Scattering terms

The inclusion of interactions between the condensate and non-condensate band give us dissipative and noise terms, and can be considered physically as growth and scattering processes (described in figure 3.2).

Growth Processes

The second line of the SPGPE (3.27b) is responsible for growth in the condensate band by the scattering of two thermal atoms in the non-condensate band. The chemical potential μ and temperature T categorize the thermal reservoir containing the atoms in the non-condensate band. The growth term $G(\mathbf{x})$ represents the number-changing collision rate of the particles in the non-condensate band and acts as a damping term in the equation of motion. The noise associated with the growth process is $dW_G(\mathbf{x},t)$, defined by

$$\langle dW_G^*(\mathbf{x},t)dW_G(\mathbf{x}',t)\rangle = 2G(\mathbf{x})\delta_{\mathbf{C}}(\mathbf{x},\mathbf{x}')dt, \tag{3.28}$$

$$\langle dW_G(\mathbf{x},t)dW_G(\mathbf{x}',t)\rangle = \langle dW_G^*(\mathbf{x},t)dW_G^*(\mathbf{x}',t)\rangle = 0, \tag{3.29}$$

where the noise is complex and additive.

The growth rate can be calculated for the non-condensate band in thermal equilibrium at chemical potential μ and temperature T, and takes the form [12]

$$G(\mathbf{x}) = \gamma,$$

with

$$\gamma = \gamma_0 \left\{ \left[\ln\left(1 - e^{\beta(\mu-\epsilon_{\mathrm{cut}})}\right)\right]^2 + e^{2\beta(\mu-\epsilon_{\mathrm{cut}})} \sum_{r=1}^{\infty} e^{r\beta(\mu-2\epsilon_{\mathrm{cut}})} \left(\Phi[e^{\beta(\mu-\epsilon_{\mathrm{cut}})}, 1, r+1]\right)^2 \right\}, \tag{3.30}$$

$\gamma_0 = 4m(ak_BT)^2/\pi\hbar^3$, and $\Phi[z,s,a]$ the Lerch transcendent

$$\Phi[z,s,a] = \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s}. \tag{3.31}$$

Scattering Processes

The third line of the SPGPE (3.27c) is responsible for the number conserving scattering process. The condensate and non-condensate atoms collide and exchange energy but the energy change is such that there is no net change in the population of each band. This term is related to the divergence of the condensate band current

$$\mathbf{j}_{\mathbf{C}} \equiv \frac{i\hbar}{2m}\left\{[\nabla\psi_{\mathbf{C}}^*(\mathbf{x})]\psi_{\mathbf{C}}(\mathbf{x}) - \psi_{\mathbf{C}}^*(\mathbf{x})\nabla\psi_{\mathbf{C}}(\mathbf{x})\right\}.$$

For the scattering process, the noise is real and defined by

$$\langle dW_M(\mathbf{x},t)dW_M(\mathbf{x}',t)\rangle = 2M(\mathbf{x}-\mathbf{x}')\mathrm{d}t,$$

where $M(\mathbf{x}-\mathbf{x}')$ is the scattering rate function.

3.9.4 Simple Growth SGPE

The scattering term (3.27c) describes the collisions between atoms in the non-condensate band and atoms in the condensate band which do not result in population changes of the two bands. This process only involves the exchange of energy and not particles, so it would seem quite reasonable that this term has far less influence on the dynamics of the BEC. This is certainly true for quasi-static systems. Because of the greater importance of the growth term in comparison to the scattering term we can justify dropping the scattering term. This is also convenient for analytic and numerical approximations to the SPGPE. The growth term is a relatively simple stochastic equation with dissipation and additive noise. The scattering term on the other hand involves an integral over the scattering rate function multiplied with the divergence of the condensate band current. The scattering term also has a multiplicative noise term. These terms are much harder to deal with analytically in terms of our variational theory.

We can treat the growth function $G(\mathbf{x})$ as constant in space if we neglect scattering and the resulting simplified version of the SPGPE is known as the simplified Growth SPGPE,

$$\begin{aligned}(S)\mathrm{d}\psi_{\mathbf{C}}(\mathbf{x},t) = \mathcal{P}_{\mathbf{C}}\Bigg\{&-\frac{i}{\hbar}\left(H_{\mathrm{sp}} + g|\psi_{\mathbf{C}}(\mathbf{x},t)|^2\right)\psi_{\mathbf{C}}(\mathbf{x})\mathrm{d}t \\ &+\frac{\gamma}{k_B T}\left[\mu - \left(H_{\mathrm{sp}} + g|\psi_{\mathbf{C}}(\mathbf{x},t)|^2\right)\right]\psi_{\mathbf{C}}(\mathbf{x})\mathrm{d}t + dW_{\gamma}(\mathbf{x},t)\Bigg\},\end{aligned} \tag{3.32}$$

with

$$\langle dW_{\gamma}^{*}(\mathbf{x},t)dW_{\gamma}(\mathbf{x}',t)\rangle = 2\gamma\delta_{\mathbf{C}}(\mathbf{x},\mathbf{x}')\mathrm{d}t.$$

Now we have achieved our goal of describing a warm Bose-Einstein condensate in terms of a stochastic differential equation. We have found that the simple growth SPGPE (3.32) (with γ determined by (3.30)) is simply the GPE with dissipation and an additive noise term. We will be able to treat the dissipation and noise as a perturbation of the GPE and use a variational approach to produce an approximate solution for the dynamics of a soliton in a warm Bose-Einstein condensate. The fact that the noise is additive allows us treat the dissipation and the noise as mathematically separate perturbations. In reality the noise is dependent on the dissipation and cannot exist without it under our formalism. The strength of the noise, given by the correlation, is related to the dissipation in accordance to the Fluctuation-Dissipation theorem. This theorem tells us the strength of the stochastic noise ensuring that the system will evolve to the correct equilibrium state. The Fluctuation-Dissipation theorem is very general and is required to guarantee the existence of steady state solutions [24].

Chapter 4

Variational theory

Here we develop a perturbative variational technique. The goal is to describe a warm Bose-Einstein condensate in terms of a perturbed GPE. From this perturbed GPE we can give an approximate dark soliton solution. The variational technique is based on the Lagrangian variational principle. We guess the form of the solution with a number of collective variables and minimize the action in terms of these. This gives us equations of motion for each of the collective variables, which we can solve explicitly to find a variational solution. In the soliton case, the basic set of collective variables describe; velocity, width, depth and position. Effects such as sound emission could be incorporated within this scheme. Ultimately we would like to use a Soliton variational ansatz to solve the SGPE for a quasi one dimensional system. This will give us the tools to study the decay of dark solitons in a finite temperature BEC. A time dependent velocity gives us the ability to create a decaying dark soliton. Using a Lagrangian variational process to determine the time dependence of the velocity will give us a theoretical description of the decay, which we can compare to numerical methods.

4.1 Classical Lagrangian mechanics

The orbits of a classical system can be described by a set of time-dependent generalized coordinates $q_1(t), \ldots, q_N(t)$, where the number of coordinates depends on the degrees of freedom in question. For a constrained system this will eliminate parameters leaving only independent coordinates. A Lagrangian, $L(q_i, \dot{q}_i, t)$ dependent on $q_1(t), \ldots, q_N(t)$, and the associated velocities, $\dot{q}_1(t), \ldots, \dot{q}_N(t)$, governs the dynamics. The Lagrangian is at most quadratic in $\dot{q}_i$ and in classical mechanics is usually given by

$$L = T - V(q_i) = \frac{1}{2} m \sum_i \dot{q}_i{}^2 - V(q_i),$$

the difference between the kinetic and potential energy of the system. The time integral

$$\mathcal{A}[q_i] = \int_{t_a}^{t_b} \mathrm{d}t L(q_i, \dot{q}_i, t),$$

of the Lagrangian along an arbitrary path $q_i(t)$ is called the *action* of the path. The Lagrangian formulation is based on *principle of least action*; which states that of all the possible paths, the classical path is the one which minimizes the action. Formally the classical path is the extremization of the action of all paths with the same endpoints. The *variation* of the action is expressed as the linear term in the Taylor expansion of $\mathcal{A}[q_i]$ in powers of $\delta q_i(t)$:

$$\delta\mathcal{A}[q_i] \equiv \{\mathcal{A}[q_i+\delta q_i] - \mathcal{A}[q_i]\}_{\mathrm{lin}}.$$

The extremal principle then becomes

$$\delta\mathcal{A}[q_i]\big|_{q_i(t)=q_i^{\mathrm{cl}}(t)} = 0,$$

for all variation around the classical path. $\delta q_i(t) \equiv q_i(t) - q_i^{\mathrm{cl}}(t)$ vanish at the endpoints and satisfy

$$\delta q_i(t_a) = \delta q_i(t_b) = 0. \tag{4.1}$$

This requirement gives,

$$\begin{aligned}
\delta\mathcal{A}[q_i] &= \{\mathcal{A}[q_i+\delta q_i] - \mathcal{A}[q_i]\}_{\mathrm{lin}} \\
&= \int_{t_a}^{t_b} \mathrm{d}t \{L(q_i(t)+\delta q_i(t), \dot{q}_i(t)+\delta\dot{q}_i(t), t) - L(q_i(t), \dot{q}_i(t), t)\}_{\mathrm{lin}} \\
&= \int_{t_a}^{t_b} \mathrm{d}t \left\{\frac{\partial L}{\partial q_i}\delta q_i(t) + \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i(t)\right\} \\
&= \int_{t_a}^{t_b} \mathrm{d}t \left\{\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right\}\delta q_i(t) + \frac{\partial L}{\partial \dot{q}_i}\delta q_i(t)\Big|_{t_a}^{t_b} = 0.
\end{aligned}$$

The boundary term disappears due to (4.1), giving the *Euler-Lagrange equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}. \tag{4.2}$$

4.1.1 Classical Hamiltonian mechanics

Alternatively things can be formulated in terms of the *Hamiltonian*.

$$H \equiv \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L(q_i, \dot{q}_i, t).$$

The value, H, is equal to the energy of the system. The natural variables of a Legendre transform are q_i and the generalized momenta p_i, where

$$p_i \equiv \frac{\partial}{\partial \dot{q}_i} L(q_i, \dot{q}_i, t).$$

We then write

$$H(p_i, q_i, t) = p_i\dot{q}_i(p_i, q_i, t) - L(q_i, \dot{q}_i(p_i, q_i, t), t).$$

It is quite often simpler to formulate the motion of a system in terms of the Hamiltonian function. The Hamiltonian is related to the Lagrangian via the Legendre transform.

The Legendre transform is its own inverse and is used in this context to transform coordinates from $\dot{q}$ to p

$$H(p,q,t) \Longleftrightarrow L(\dot{q},q,t), \quad p = \frac{\partial L}{\partial \dot{q}}.$$

This can only be done if H or equivalently L are convex. If not then we will not be able to find a Lagrangian associated with a given Hamiltonian.

In terms of the Hamiltonian the action is

$$\mathcal{A}[p_i,q_i] = \int_{t_a}^{t_b} \mathrm{d}t \sum_i [p_i(t)\dot{q}_i(t) - H(p_i(t),q_i(t),t)].$$

Looking at the variation

$$\begin{aligned}
\delta\mathcal{A}[p_i,q_i] &= \int_{t_a}^{t_b} \mathrm{d}t \sum_i \left[\delta p_i(t)\dot{q}_i(t) + p_i(t)\delta\dot{q}_i(t) - \frac{\partial H}{\partial p_i}\delta p_i - \frac{\partial H}{\partial q_i}\delta q_i\right] \\
&= \int_{t_a}^{t_b} \mathrm{d}t \sum_i \left\{\left[\dot{q}_i(t) - \frac{\partial H}{\partial p_i}\right]\delta p_i - \left[\dot{p}_i(t) + \sum_i \frac{\partial H}{\partial q_i}\right]\delta q_i\right\} \\
&+ p_i(t)\delta q_i(t)\big|_{t_a}^{t_b}.
\end{aligned}$$

The boundary term disappears and the variation must be zero so we get the *Hamilton equations* of motion

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}. \tag{4.3}$$

The phase space variables satisfy several identities using *Poisson brackets*,

$$\{A,B\} \equiv \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} - \frac{\partial B}{\partial p_i}\frac{\partial A}{\partial q_i}.$$

We have that

$$\{p_i,q_j\} = \delta_{ij}, \quad \{p_i,p_j\} = \{q_i,q_j\} = 0.$$

The Poisson brackets in classical mechanics play the role of the commutator relations in quantum mechanics.

4.1.2 Non-Conservative Forces

We can quite easily see that by setting

$$L = T - V, \quad T = \frac{1}{2}mv^2,$$

we recover Newtons equations. So to phrase a problem in classical mechanics in terms of the Lagrangian approach we need simply identify the kinetic and potential terms of the system and produce our Lagrangian, L.

For conservative forces one can always write $F = -\nabla V$, so that the potential is dependent only on the final position and not on the path taken.

$$T = \frac{1}{2} m \sum_i \dot{q}_i^2,$$
$$\frac{\partial L}{\partial q_i} = -\frac{\partial V}{\partial q_i} = F_i \quad \text{The } i^{th} \text{ component of the force}$$
$$\frac{\partial L}{\partial \dot{q}_i} = m\dot{q}_i \equiv p_i \quad \text{The } i^{th} \text{ component of momentum}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad \Rightarrow \quad m\ddot{q}_i = \frac{dp_i}{dt} = F_i \quad \text{Newtons second Law}$$

Non-conservative forces Q_i cannot be written as the gradient of some scalar potential and hence cannot be absorbed into the potential term V. In this case

$$m\ddot{q}_i - F_i = Q_i \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i. \tag{4.4}$$

The effects of non-conservative forces arise as a non-zero right hand side of the Euler-Lagrange equations. This idea is important and we will find a similar type of adjustment is required when treating perturbed soliton equations of motion in section 4.4.

4.2 Lagrangian Field theory

As we have seen in section 3.9.4, the warm Bose-Einstein condensate description can be described in terms of a differential equation for a c-field function $\psi_{\mathbf{C}}(\mathbf{x}, t)$. The Lagrangian variational technique still holds, but must be phrased in the form of Lagrangian Field theory. When moving to continuous fields, such as electromagnetic fields or quantum field theory, we find that the principle of least action and hence the Lagrangian technique still holds with only minimal differences. Unlike the discrete classical mechanics case where we were describing a finite number of discrete particles, the field is continuous in the spatial dimensions as well as time. The *action* is defined as

$$S[\phi] = \int_{\Omega_4} d^4x \mathcal{L}, \tag{4.5}$$

where $\mathcal{L}$ is the *Lagrangian density.* We usually consider the time dimension separately and wish to minimize the path with respect to time. In this case we minimize $S[\phi]$,

$$S[\phi] = \int_{t_0}^{t_1} L dt, \quad L = \int_{\Omega_3} d^3x \mathcal{L},$$

where Ω_3 is the spatial volume. We define $\langle L \rangle$ as

$$\langle L \rangle = \int dt L,$$

the integral over both the spatial and time dimensions. The form of Eq.(4.5) shows how things can be formalized to be compatible with Special and General relativity, showing the Lagrangian principle is fundamental to physics. For a set of coordinates described by x^μ, the Lagrangian field density is usually just a function of ϕ and its first order derivative $\partial_\mu \phi$. Considering

$$S[\phi + \delta\phi] = \int_{\Omega_4} d^4x \mathcal{L}(\phi + \delta\phi, \partial_\mu(\phi + \delta\phi)),$$

noting that $\partial_\mu(\delta\phi) = \delta\partial_\mu\phi$; we find that setting $\delta S = 0$ gives the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right), \tag{4.6}$$

in a process very similar to the classical case. The notable difference is that the derivatives are in terms of the field ϕ instead of the generalized coordinates. We are usually concerned with the variation in time where $x^\mu = t$ and the field version of the Euler-Lagrange equations become

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = 0. \tag{4.7}$$

The difficulty of setting up a problem in the Lagrangian field formalism is determining the field density $\mathcal{L}$ associated with the equation of motion describing the system. In contrast the classical case we do not have an algorithmic process to derive an expression for the Lagrangian density for an arbitrary system. In fact there are infinitely many partial differential equations which do not have a corresponding Lagrangian density. This is not just a mathematical curiosity there are physical systems which do not have Lagrangians associated with them. It is usually easier to describe the system in terms of a Hamiltonian and perform a variational approach much the same as for the Lagrangian case. The Lagrangian when it exists can be found from the Hamiltonian (and vice versa) using the Legendre transform.

As a concrete and relevant example of a Lagrangian density and its correspondence to an equation of motion consider

$$\mathcal{L} = \frac{i}{2} \left(\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t} \right) + \frac{1}{2} \left| \frac{\partial \psi}{\partial x} \right|^2 + \frac{1}{2} |\psi|^4. \tag{4.8}$$

We solve

$$\delta \int_{t_0}^{t_1} \int_{-\infty}^{\infty} \mathcal{L} \left(\psi, \psi^*, \frac{\partial \psi}{\partial x}, \frac{\partial \psi^*}{\partial x}, \frac{\partial \psi}{\partial t}, \frac{\partial \psi^*}{\partial t} \right) dx dt = 0,$$

where ψ, ψ^* are independent.

$$\delta \int L dt = 0 \Rightarrow \frac{\delta \langle L \rangle}{\delta \psi^*} = 0, \quad \frac{\delta \langle L \rangle}{\delta \psi} = 0.$$

From $\delta\langle L \rangle / \delta\psi^* = 0$, we get

$$i \frac{\partial \psi}{\partial t} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} - |\psi|^2 \psi = 0. \tag{4.9}$$

This is the one dimensional Nonlinear Schrödinger equation (NLS). The other equation $\frac{\delta \langle L \rangle}{\delta \psi} = 0$, gives the complex conjugate of the NLS. We can see that the solution set associated to minimizing the action of the Lagrangian density Eq.(4.8) is identical to the solutions of the NLS equation. One might ask what boundary conditions we are using to solve the NLS equation if we use the variational method. We require that the integrals over space converge when calculating L from $\mathcal{L}$. This requires that $\psi \to 0$ as $x \to \pm\infty$ and establishes our boundary conditions. The Lagrangian (4.8) is used as the base for the variational method by Anderson, for creating approximate gaussian solutions in optics problems that are described by the NLS equation (For details see [4]). The Lagrangian (4.8) represents a material with Kerr non-linearity in optics, a Bose-Einstein condensate, or any other system described by the non-linear Shrödinger equation.

4.3 Symmetries and Conservation

Symmetries play a very fundamental and important role in physics. The appearance of symmetries is related to group theory and the existence of a particular symmetry has many important implications; creating conserved quantities and placing strong restrictions on the forms of equations and solutions. The study of symmetries in a system gives us a lot of non-trivial information about the system. Symmetry is a very important aspect of field theory, the universe we live in exhibits a number of symmetries and invariant transformations. A correct theoretical description must exhibit these symmetries. This section is quite technical but the main point is that we can use the symmetries of our system to predict the form of conserved quantities, and pick a Lagrangian density which is consistent with these requirements. This will give us the correct Lagrangian for our dark soliton described by the NLS equation, giving us a base for our perturbative methods.

4.3.1 Cyclic Coordinates and Conservation

A coordinate q_i is said to be *cyclic* if the Lagrangian L does not depend on t,

$$\frac{dL}{dt} = 0 \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial \dot{q}_i} = \text{const.}$$

So a cyclic coordinate implies a conserved quantity. We can look at conserved quantities from a more general point of view. In Lagrangian field theory we use a continuous variable ϕ instead of discrete coordinates q_i. In this case we are looking at continuous symmetries of the system and their implications for conserved quantities. Translations in the independent dimensions are created by infinitesimal generators δx_μ. There are also generators associated with rotations. Nöthers theorem is a statement relating the symmetries of the Lagrangian (or Hamiltonian) with conserved quantities. The theorem is very general but is most useful in the context of a four-space classical field theory. Formally Referring to [29] we consider an infinitesimal transformation of the form

$$x_\mu \to x'_\mu = x_\mu + \delta x_\mu,$$

where the infinitesimal change δx_μ may be a function of all the other x_ν. Nöthers theorem also considers the effect of a transformation in the fields (n component field ϕ_ρ, $\rho = 0, 1, ..., n-1$) themselves,

$$\phi_\rho(x_\mu) \to \phi'_\rho(x'_\mu) = \phi_\rho(x_\mu) + \delta\phi_\rho(x_\mu),$$

where $\delta\phi_\rho(x_\mu)$ measures both the change in x_μ and ϕ_ρ and may be a function of all other field quantities ϕ_λ. Note that we are dealing with translational type symmetries and not parity changes. Nöthers theorem states that if the Lagrangian density displays the same functional form in terms of transformed quantities as it does of the original quantities

$$\mathcal{L}'(\phi'_\rho(x'_\mu), \phi'_{\rho,\nu}(x'_\mu), x'_\mu) = \mathcal{L}(\phi_\rho(x_\mu), \phi_{\rho,\nu}(x_\mu), x_\mu),$$

and the magnitude of the action integral is invariant under the transformation, then there is a conserved quantity for every symmetry. The theorem states that every symmetry induces a conserved quantity. Moreover, Nöthers theorem gives the explicit form of these conserved quantities (4.10). There may however, be conserved quantities which do not correspond to a symmetry property of the Lagrangian. For example using Inverse Transform Methods it is possible to show the the NLS equation (4.9) has an infinite number of conserved quantities.

Returning to the Lagrangian Eq.(4.8) we note that it does not explicitly depend on either t or x, and thus there must be two conserved quantities associated with these translational symmetries. In this case we have (from [47]) the conserved quantity

$$G\{u\} = -\int dx \left\{ \mathcal{L}\delta t + \left[\frac{\partial \mathcal{L}}{\partial(\partial u/\partial t)} \left(-\delta t \frac{\partial u}{\partial z} - \delta x \frac{\partial u}{\partial x} + \mathbf{c.c} \right) \right] \right\}, \tag{4.10}$$

where **c.c** stands for complex conjugate. We can separate the terms proportional to δt and δx to get,

$$E = -\int dx \left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial u/\partial t)} \frac{\partial u}{\partial t} - \frac{\partial \mathcal{L}}{\partial(\partial u^*/\partial t)} \frac{\partial u^*}{\partial t} \right) = \int dx \left(\frac{1}{2} \left| \frac{\partial u}{\partial x} \right|^2 + \frac{1}{2} |u|^4 \right),$$

and

$$I = \int \left(\frac{\partial \mathcal{L}}{\partial(\partial u/\partial t)} \frac{\partial u}{\partial x} + \frac{\partial \mathcal{L}}{\partial(\partial u^*/\partial t)} \frac{\partial u^*}{\partial x} \right) dx = \int dx \frac{i}{2} \left(u \frac{\partial u^*}{\partial x} - u^* \frac{\partial u}{\partial x} \right),$$

representing energy and momentum respectively. We also have another physically relevant invariant quantity, the conservation of power or the normalization condition

$$P = \int dx |u|^2.$$

This conserved quantity is not a consequence of the symmetry properties but rather the fact that the time evolution operator in quantum mechanics is unitary.

4.3.2 Gaussian Ansatz examples

The Lagrangian Eq.(4.8) discussed above allows us to solve the NLS equation via the variational principle. As a related example Anderson [4] used the Lagrangian variational approach to solve a nonlinear optics problem of wave propagation, described by

$$i\frac{\partial \psi}{\partial x} = \alpha \frac{\partial^2 \psi}{\partial \tau^2} + \kappa |\psi|^2 \psi. \tag{4.11}$$

The initial pulse is assumed to be well approximated by the function

$$\psi(0,\tau) = A_0 \exp\left(-\frac{\tau^2}{2a_0^2}\right),$$

where A_0 is the maximum amplitude of the pulse and a_0, the width. We assume that the waveform will be described by a function of the form,

$$\psi(\tau,x) = A(x)\exp\left(-\frac{\tau^2}{2a^2(x)} + ib(x)\tau^2\right). \tag{4.12}$$

This trial function is substituted into the Lagrangian density and a new Lagrangian density is found in terms of the collective variables $a(x), b(x)$ and $A(x)$. The Lagrangian density is given by

$$\mathcal{L} = \frac{i}{2}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right) - \alpha\left|\frac{\partial\psi}{\partial\tau}\right|^2 + \frac{\kappa}{2}|\psi|^4,$$

which creates the Gaussian ansatz Lagrangian

$$L_G = \frac{\sqrt{\pi}}{2}\left[ia\left(A\frac{\partial A^*}{\partial x} - A^*\frac{\partial A}{\partial x}\right) + |A|^2a^3\frac{\partial b}{\partial x} - a^3\alpha|A|^2\left(\frac{1}{a^4} + 4b^2\right) + \frac{1}{\sqrt{2}}\kappa a|A|^4\right].$$

We then minimize the action by varying the collective variables and get a set of equations corresponding to $\delta\int L\,dx = 0$

$$\frac{\delta\int L_G}{\delta A^*} = 0, \;\; \frac{\delta\int L_G}{\delta A} = 0, \;\; \frac{\delta\int L_G}{\delta a} = 0, \;\; \frac{\delta\int L_G}{\delta b} = 0.$$

These simultaneous equations can be solved to give

$$a|A|^2 = \text{const} \Rightarrow \int_{-\infty}^{\infty}|\psi|^2 dx = \text{const},$$

$$\frac{da}{dx} = -4a\alpha b,$$

$$\frac{d^2a}{dx^2} = -\frac{d}{da}V(a), \quad V(a) = \frac{2\alpha^2}{a^2} - \frac{\alpha\kappa\sqrt{2}E_0}{a} + c.$$

Thus, the equation of motion describing a, is that of a particle in a potential. Once we have determined a we can get b, A, A^*.

This variational approach allows us to get relatively simple equations of motion for the functions considered. Here $a(x)$ describes the width and $A(x)$ the amplitude. For example, looking at the equations of motion for a, we can see how the width of the gaussian changes and gain insight to the approximate evolution. This is the power of the variational method, the ability to simplify the problem of evolution and look at individual properties, such as width or amplitude, separately. The cost of this is the fact we are settling for approximate solutions.

By using a different trial function we will get a different set of equations. In [20], Duine and Stoof found a variational solution to the GPE using the Gaussian function

$$\psi = \sqrt{N_c(t)}e^{i\theta_0(t)}\left(\frac{1}{\pi q^2(t)}\right)^{1/4}\exp\left(-\frac{x^2}{2q^2}(1 - \frac{im}{\hbar}q(t)\dot{q}(t))\right).$$

This is equivalent to solving the Lagrangian problem

$$\delta \iint \mathcal{L} dx dt = 0, \tag{4.13}$$

where

$$\mathcal{L} = \frac{i}{2}\hbar\left(\psi\frac{\partial\psi^*}{\partial t} - \psi^*\frac{\partial\psi}{\partial t}\right) + \frac{\hbar^2}{2m}\left|\frac{\partial\psi}{\partial x}\right|^2 + V(x)|\psi|^2 + \frac{2\pi\hbar^2 a}{m}|\psi|^4.$$

Setting the variation of each free function to zero

$$\frac{\delta\int L_G}{\delta N} = 0,\ \frac{\delta\int L_G}{\delta\theta} = 0, \frac{\delta\int L_G}{\delta q} = 0$$

we can derive a set of equations which can be solved simultaneously yielding equations analogous to that of a particle in a well.

$$\frac{1}{2}N_c m\ddot{q} = -\frac{\partial}{\partial q}V(q(t), N_c(t)),$$

$$V(q(t), N_c(t)) = \frac{1}{4}N_c m\omega^2 q^2 + \frac{N_c\hbar^2}{4mq^2} + \frac{\sqrt{2\pi}a\hbar^2 N_c^2}{mq}.$$

We find $N = \text{const}$, (the condensate does not grow or evaporate for standard GPE) and an expression for θ in terms of N and q. Duine and Stoof have found a more complete and accurate solution than the first ansatz by Anderson but at the expense of more complicated calculations.

4.3.3 Soliton renormalized Integrals

When applying the same method to dark solitons one very quickly runs into problems. The Gaussian functions (and bright soliton functions) decay to zero as $x \to \pm\infty$ sufficiently fast that to produce finite values for E, I and P. The dark soliton describes a dip in the density of a homogenous background. When considering an infinite system the number of particles becomes infinite and the energy, momentum and power integrals all diverge. In order to apply the Lagrangian approach me must either consider an explicitly finite system, or renormalize our equations. We follow Kivshar and Królikowski [47] and subtract out the effects of the constant background. This adjustment of the NLS equation is simple, but the effect on the Lagrangian density $\mathcal{L}$ is more subtle. Assuming that $|u_0|^2 = 1$, we are now solving the renormalized Nonlinear Schrödinger equation,

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - (|u|^2 - 1)u = 0. \tag{4.14}$$

Comparing Eq.(4.14) with Eq.(4.9) one might naively assume that the correct Lagrangian would simply be

$$\mathcal{L} = \frac{i}{2}\left(u^*\frac{\partial u}{\partial t} - u\frac{\partial u^*}{\partial t}\right) - \frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2 - \frac{1}{2}(|u|^2 - 1)^2,$$

where we have made the substitution $|u|^2 :\to (|u|^2 - 1)$. This Lagrangian density has been assumed by a number of people in many papers (for example [34] and most papers before

1994), solves the renormalized NLS but is incorrect. To find the correct Lagrangian we must renormalize the energy, momentum and power. We appeal to the Nöthers theorem to establish the correct form. To renormalize the integrals we wish to calculate the contribution from just the soliton by subtracting off the component due to the homogenous background. Again assuming that $u_0^2 = 1$, we want to calculate the energy, Hamiltonian and momentum of the soliton itself. The power;

$$P_s = \int_{-\infty}^{\infty} dx (u_0^2 - |u|^2), \tag{4.15}$$

Hamiltonian

$$E_s = \frac{1}{2} \int_{-\infty}^{\infty} dx \left(\left| \frac{\partial u}{\partial x} \right|^2 + (|u|^2 - u_0^2)^2 \right), \tag{4.16}$$

and momentum

$$I_s = \frac{i}{2} \int_{-\infty}^{\infty} dx \left(u \frac{\partial u^*}{\partial x} - u^* \frac{\partial u}{\partial x} \right) - \arg u|_{-\infty}^{\infty},$$

where $\arg u|_{-\infty}^{\infty}$ the contribution of the continuous wave phase which varies continously through the soliton.

$$\arg u|_{-\infty}^{\infty} = \int_{-\infty}^{\infty} dx \frac{\partial}{\partial x} \tan^{-1} \left(\frac{\Im(u)}{\Re(u)} \right) = \frac{1}{|u|^2} \frac{i}{2} \left(u \frac{\partial u^*}{\partial x} - u^* \frac{\partial u}{\partial x} \right).$$

So that

$$I_s = \frac{i}{2} \int_{-\infty}^{\infty} dx \left(u \frac{\partial u^*}{\partial x} - u^* \frac{\partial u}{\partial x} \right) \left(1 - \frac{1}{|u|^2} \right). \tag{4.17}$$

From the definition

$$I_s = \int dx \left(\frac{\partial u}{\partial x} \frac{\partial \mathcal{L}}{\partial(\partial u/\partial z)} + \frac{\partial u^*}{\partial x} \frac{\partial \mathcal{L}}{\partial(\partial u^*/\partial z)} \right),$$

we find that the correct renormalized Lagrangian density is

$$\mathcal{L}_s u = \frac{i}{2} \left(u^* \frac{\partial u}{\partial t} - u \frac{\partial u^*}{\partial t} \right) \left(1 - \frac{1}{|u|^2} \right) - \frac{1}{2} \left| \frac{\partial u}{\partial x} \right|^2 - \frac{1}{2} (|u|^2 - 1)^2. \tag{4.18}$$

The extra factor $(1 - 1/|u|^2)$ gives a difference between the regular Lagrangian of $\frac{d}{dz} \log(u/u^*)$ which does not change the motion equation and thus $\mathcal{L}_s$ satisfies the renormalized NLS equation Eq.(4.14). This is the correct Lagrangian to use when considering a dark soliton and will have important consequences in terms of convergence when you try to calculate the Lagrangian from the Lagrangian density.

With a little analysis of the symmetry properties the correct Lagrangian density, $\mathcal{L}$, was established for a dark soliton in a homogenous Bose-Einstein condensate described by the NLS equation. The adjustment of $\mathcal{L}$ by subtracting off the background to give the correct energy and momentum for the soliton was subtle. However, now this has been established once and for all, we can substitute any dark soliton ansatz into it and carry out the Lagrangian variational approach.

4.4 Perturbation Theory

The only exactly integrable NLS-type equation is for the purely cubic nonlinearity. It is only for exactly integrable differential equations that we have the exact soliton solutions. We have exact soliton solutions for the idealized NLS equation

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - (|u|^2 - 1)u = 0,$$

but would like to solve the physically more interesting case where this equation is modified by damping and stochastic noise. We wish to solve the SGPE for a soliton at a finite temperature. One way to deal with this situation is to consider the perturbed equation

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - (|u|^2 - 1)u = \epsilon R(u), \tag{4.19}$$

where $\epsilon R(u)$ is our *perturbation*, which includes the finite temperature effects. If we assume that the perturbation is small then Eq.(4.19) is a *nearly integrable* differential equation. We would imagine that the solutions of the perturbed equation, while not being true soliton solutions would look very similar to the corresponding soliton solutions for the exactly integrable equation. This is the basis of the perturbation technique in terms of the Lagrangian approach. To solve a perturbed equation we can use as an initial ansatz function the exact solution of the corresponding unperturbed case and let the collective variables such as depth, width and velocity become time dependent. The Lagrangian approach will give equations of motion for these parameters. This is the approach used in my project, where the perturbation will include a damping term and a stochastic noise term.

The importance of perturbation methods is the inclusion of non-conservative forces. As mentioned previously any conservative force can be written as the gradient of a scalar potential, so can be included in the potential terms in the Lagrangian density. Non-conservative forces cannot in general be written this way and have to be included in some other fashion. What we would like to do is relate the Lagrangian equations of motion to the Perturbation term in Eq.(4.19). The non-conservative force will mean that the conserved quantities of the unperturbed equation are no longer conserved. If for example we have a dissipative term, then the energy of the system will decrease and the rate of change of this decrease will be related to the strength of the dissipation i.e. the perturbation term. In fact it is possible to derive adjusted equations of motion by considering the term $\frac{\partial E}{\partial t}$ and its dependence on $\epsilon R(u)$ (see [34]). For a one dimensional Lagrangian which is dependent on the solution parameters $a_j(t)$, so that $u(t, x, a_j(t))$, we have the general result that (cf. Eq.(4.4))

$$\frac{\partial L}{\partial a_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}_j}\right) = Q,$$

where Q is dependent on $\epsilon R(u)$.

For a Lagrangian $L(a_j)$ we can perform the standard procedure of minimizing the action, S, with respect to the time coordinate and find

$$\frac{\partial L}{\partial a_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}_j}\right) = \delta S, \tag{4.20}$$

we can also consider the minimizing the action in terms of the Lagrangian density $\mathcal{L}$, which is given by

$$\delta S = \int_{\Omega_4} d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] \delta\phi.$$

If we consider the one-dimensional case where $\mathcal{L}(u, u_x, u_t, u^*, u_x^*, u_t^*)$ then

$$\delta S = \int dxdt \left[\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial u_t} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x} + \mathbf{c.c} \right] \delta u.$$

We also note that $u(a_j)$ so that $\delta u = \frac{\partial u}{\partial a_j}\delta a_j$. δa_j is dependent only on time. If we consider minimizing only with respect to time as in (4.20) we will be left with just a spatial integral and can equate (4.20) with the above expression

$$\frac{\partial L}{\partial a_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}_j}\right) = \int_{-\infty}^{\infty} dx \left[\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial u_t} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x} + \mathbf{c.c} \right] \frac{\partial u}{\partial a_j}.$$

This is the result quoted in [47]. Substituting Eq.(4.18) we find

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial u_t} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x} &= -i\frac{\partial u^*}{\partial t} + \frac{1}{2}\frac{\partial^2 u^*}{\partial x^2} - (|u|^2 - 1)u^* \\
&= \left(i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - (|u|^2 - 1)u \right)^* \\
&= \epsilon R^*(u).
\end{aligned}$$

So the effects of a perturbation $\epsilon R(u)$ on the equations of motion quoted from [47] is given by

$$\frac{\partial L}{\partial a_j} - \frac{d}{dz}\left(\frac{\partial L}{\partial \dot{a}_j}\right) = 2\epsilon \Re \left(\int_{-\infty}^{\infty} R^*(u) \frac{\partial u}{\partial a_j} \right). \tag{4.21}$$

This is the equation we use to describe the effects of perturbations on the dark soliton. We now know how to describe the evolution of a dark soliton in a homogenous infinite sized finite temperature BEC in the perturbation regime. Now it is simply a matter of finding the correct description of the perturbation to correspond to the SGPE.

4.4.1 Dissipation and stochastic noise

We are interested in describing a soliton within a BEC under the influence of dissipation and stochastic noise. We wish to consider the equation

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - (|u|^2 - 1)u = \gamma f(u) + \eta(x,t), \tag{4.22}$$

where γ is the strength of dissipation and $\eta(x,t)$ is a random fluctuating function. We know that the strength of the correlation $\langle \eta(x', t')^* \eta(x,t) \rangle$ must be related to the strength of dissipation so that the fluctuation-dissipation theorem holds and the system settles down to the correct equilibrium distribution. The strength of correlation should also be related to the temperature of the BEC. The values of γ and $\langle \eta(x', t')^* \eta(x,t) \rangle$ can be derived from a master equation for the one-body density matrix using second-order Perturbation theory [26] or using field theoretical techniques [42].

We can determine the strength of dissipation and noise by scaling the simple growth SGPE (3.32). If we work in one dimension, drop the explicit projection operator $\mathcal{P}_{\mathbf{C}}$ and put our wave in a rotating frame $\psi(x,t) = \psi_{\mathbf{C}}(x,t)e^{i\mu/\hbar}$, we have

$$(S)i\hbar \mathrm{d}\psi + \mu\psi \mathrm{dt} = \left(H_{\mathrm{sp}} + g|\psi|^2\right)\psi \mathrm{dt} + \frac{ih\gamma}{k_B T}\left(\mu - (H_{\mathrm{sp}} + g|\psi|^2)\right)\psi \mathrm{dt} + i\hbar dW_\gamma(x,t),$$

with

$$\langle dW_\gamma^*(x,t), dW_\gamma(x',t')\rangle = 2\gamma\delta(x-x')\delta(t-t').$$

Letting $V_{\mathrm{ext}}(x) = 0$ and rearranging we find

$$i\hbar\frac{\partial\psi}{\partial t} - \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \left(g|\psi|^2 - \mu\right)\right]\psi = -\frac{i\hbar\gamma}{k_B T}\left[\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \left(g|\psi|^2 - \mu\right)\right]\psi + i\hbar\frac{dW_\gamma(x,t)}{dt} \tag{4.23}$$

If we make the scaling

$$\chi = \frac{x}{x_c},\ x_c = \frac{\hbar}{\sqrt{m\mu}}, \qquad \tau = \frac{t}{t_c},\ t_c = \frac{\hbar}{\mu}, \qquad u = \sqrt{n}\psi,\ n = \frac{\mu}{g},$$
$$\gamma' = \frac{\hbar\gamma}{k_B T}, \qquad \xi(x,t) = i\hbar\frac{dW_\gamma(x,t)}{dt},$$

we find

$$i\frac{\partial u}{\partial\tau} + \frac{1}{2}\frac{\partial^2 u}{\partial\chi^2} - (|u|^2 - 1)u = -i\gamma'\left(-\frac{1}{2}\frac{\partial^2 u}{\partial\chi^2} + (|u|^2 - 1)u\right) + \xi(\chi,\tau), \tag{4.24}$$

with correlation,

$$\langle\xi^*(\chi,\tau)\xi(\chi',\tau')\rangle = T'\gamma'\delta(\chi'-\chi)dt', \tag{4.25}$$

where $T' = k_B T/\mu$. Thus we have an equation to describe the perturbation due to dissipation and stochastic noise in a Bose-Einstein Condensate of the form (4.19), with

$$\epsilon R(u) = -i\gamma'\left(-\frac{1}{2}\frac{\partial^2 u}{\partial x^2} + (|u|^2 - 1)u\right) + \xi(x,t). \tag{4.26}$$

The expression for γ' is given analytically by

$$\gamma' = \frac{\hbar}{k_B T}\gamma,$$

where γ is given in section 3.9.3 by (3.30) and is dependent on the temperature. In determining the dissipation we have assumed that the damping, which is proportional to $G(\mathbf{x})$, is constant. Treating $G(\mathbf{x})$ as constant is in fact a valid approximation. Bradley *et al.* has shown that for $\{x|V(\mathbf{x}) \leq 2\epsilon_{\mathrm{cut}}/3\}$ $G(\mathbf{x})$ is spatially invariant and is only weakly dependent on position for $\{2\epsilon_{\mathrm{cut}}/3 < V(\mathbf{x}) \leq E_{\max}\}$ [12]. From now on we will drop the prime and let γ be the temperature dependent rescaled damping coefficient, similarly we will drop the prime on the rescaled temperature T'. This scaling could be done equivalently via the formulation developed by Stoof [42], in terms of the Keldysh self energy $\Sigma^K(x,t)$. The Keldysh self energy function plays the role of the Growth function $G(x,t)$ introduce by Gardiner and Zoller in the formalism described in this project. We have the relationship

$$\Sigma^K(x,t) = -4iG(x,t).$$

The important point is that we can determine the appropriate strength of the dissipation and stochastic noise such that it accurately reflects theoretical descriptions and obeys the Fluctuation-Dissipation theorem. We now have an expression for dealing the effects of finite temperature and a procedure to establish the effects of dissipation and stochastic noise on the soliton.

Chapter 5

Analytic Results

In this chapter we use the results of the previous chapters to solve the variational Lagrangian based on the SGPE formalism and obtain approximate analytic solutions for soliton decay in a homogeneous infinite system.

5.1 Single soliton

5.1.1 Single Soliton under the effect of dissipation

For a homogenous BEC with zero external potential the wavefunction of a single dark soliton is

$$\psi = \sqrt{n}\left(i\frac{v}{c} + \sqrt{1-\frac{v^2}{c^2}}\tanh\left(\frac{x-vt}{\xi}\sqrt{1-\frac{v^2}{c^2}}\right)\right)e^{int},$$

where ξ is the healing length $\xi = \hbar/\sqrt{m\mu}$. In scaled coordinates and ignoring the rotating phase factor we can write the solution as

$$u(x,t) = B\tanh\left[B(x-vt)\right] + iA,$$

with $A = v$, $A^2 + B^2 = 1$.

The exact soliton dark soliton solution is a travelling wave with fixed shape, an example of a non dispersive wave. There is no exact soliton solution in the case of dissipation and noise, however we would expect that the an initial soliton waveform would retain an approximately soliton shaped. Thus we would expect that evolution of a soliton under the effects of dissipation and noise to be described well by the approximate solution

$$u(x,t) = B(t)\tanh\left[D(t)(x-x_0(t))\right] + iA(t), \quad A^2 + B^2 = 1. \tag{5.1}$$

The function $B(t)$ describes the depth of the soliton, $D(t)$ the width and $x_0(t)$ the position. We require only that A and B be related, however we will find that a relationship between the depth, width and velocity will emerge through the equations of motion. It is important to note that the functions A and B are not independent, we will not get two independent equations by considering $\frac{\partial L}{\partial A}$ and $\frac{\partial L}{\partial B}$. Instead we consider A to be a function of B and only consider $\frac{\partial L}{\partial B}$. The Lagrangian associated with this dark soliton ansatz function is

$$L = 2\frac{dx_0}{dt}\left[\tan^{-1}\left(\frac{B}{A}\right) - AB\right] - \frac{2}{3}\left(B^2 D + \frac{B^4}{D}\right). \tag{5.2}$$

As a sanity check we can solve the Euler-Lagrange equations Eq.(4.2) without the effects of dissipation and noise and we find that

$$D = B, \quad \frac{dB}{dt} = 0 \quad \text{and} \quad \frac{dx_0}{dt} = A = v,$$

recovering the exact solution in this case.

5.1.2 Dissipative acceleration

We consider the effects of the dissipative term of (4.26),

$$\epsilon R(u) = -i\gamma' \left(-\frac{1}{2}\frac{\partial^2 u}{\partial x^2} + (|u|^2 - 1)u \right). \tag{5.3}$$

This is a theoretically justified model of damping which should give a reasonably good description of dissipation in the absence of noise.

The equations for the unperturbed soliton motion, that of a soliton described by the Gross-Pitaevskii equation are well known. We now find new equations which describe the soliton in a high temperature system using our theoretically justified perturbation. Results for solitons under the effect of a number of types of dissipation have been obtained by others (see for example the review article [33]). To the best of our knowledge damping of this precise form, for which there is a microscopic theory, has not been calculated for BEC's.

We must evaluate the right hand side of Eq.(4.21),

$$\epsilon \Re \left(\int_{-\infty}^{\infty} dx R^*(u) \frac{\partial u}{\partial B} \right) = \int_{-\infty}^{\infty} dx \Re \left\{ i\gamma(-\frac{1}{2}u^*_{xx} + (|u| - 1)u^*) \frac{\partial u}{\partial B} \right\} = 0,$$

$$\epsilon \Re \left(\int_{-\infty}^{\infty} dx R^*(u) \frac{\partial u}{\partial x_0} \right) = -\frac{4\gamma A B^3}{3},$$

$$\epsilon \Re \left(\int_{-\infty}^{\infty} dx R^*(u) \frac{\partial u}{\partial D} \right) = 0.$$

The effects of dissipation only affect the equation of motion for the position of the operator. However due to the relationship between velocity and depth, we will see an effect on the amplitude.

The equations of motion for a single soliton in our dissipative system are defined through

$$\frac{\partial L}{\partial B} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{B}} \right) = 2\epsilon \Re \left(\int_{-\infty}^{\infty} dx R^*(u) \frac{\partial u}{\partial B} \right),$$

$$4\dot{x}_0 \frac{B^2}{A} - \frac{4}{3}\left(BD + \frac{2B^3}{D} \right) = 0. \tag{5.4}$$

$$\frac{\partial L}{\partial x_0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_0} \right) = 2\epsilon \Re \left(\int_{-\infty}^{\infty} dx R^*(u) \frac{\partial u}{\partial x_0} \right),$$

$$4\dot{A}B = \frac{8\gamma A B^3}{3}. \tag{5.5}$$

$$\frac{\partial L}{\partial D} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{D}}\right) = 2\epsilon \Re \left(\int_{-\infty}^{\infty} dx R^*(u) \frac{\partial u}{\partial D} \right)$$
$$- \frac{2}{3}\left(B^2 - \frac{B^4}{D^2}\right) = 0. \tag{5.6}$$

Eq.(5.6) implies that $B^2 = D^2$ and Eq.(5.4) implies that $\frac{dx_0}{dt} = \pm A$, with $+A$ corresponding to $B = D$ and $-A$ for $B = -D$. The two solutions correspond to propagation in the positive or negative directions respectively. These relationships hold just like the unperturbed case. The new condition is that of Eq.(5.5), which using the identity $\dot{A} = -B\dot{B}/A$, gives

$$\dot{B} = -\frac{2\gamma}{3} B(1 - B^2) \quad \Rightarrow \quad \frac{B}{A} = \frac{B_0}{A_0} e^{-\frac{2\gamma}{3}t}.$$

The equation tells us that as $t \to \infty$, $B/A \to 0$ exponentially so that $B \to 0$. This implies that $A \to c$, the speed of sound in the condensate ($c = 1$ in our scaling). The Effect of dissipation causes decay of the soliton amplitude. The connection between amplitude and velocity causes the soliton accelerate as it decays.

The wavefunction is given by

$$u = \sqrt{1 - v^2}\tanh(\sqrt{1 - v^2}(x - v)) + iv,$$

for

$$\frac{\sqrt{1 - v^2}}{v} = \frac{\sqrt{1 - v_0^2}}{v_0} e^{-\frac{2\gamma}{3}t},$$

or equivalently

$$v = \frac{v_0}{\sqrt{(1 - v_0^2)e^{-\frac{4\gamma}{3}t} + v_0^2}}. \tag{5.7}$$

The resulting velocity for a number of dissipation rates is shown in figure 5.1.

This damping induced acceleration is a counterintuitive result. Ordinarily we associate damping with a decrease in velocity (and decay of amplitude when considering some sort of disturbance). The acceleration is caused by the fact that the velocity of a soliton is coupled to the depth. The dissipative forces acting on the soliton causes it to lose energy which causes the disturbance (density dip) to decrease. Damping induced acceleration is surprising but does make sense. Due to the localization of the soliton we can consider it a particle. In this description the dark soliton is a particle with negative effective mass.

The fact that solitons accelerate under the effects of dissipation is well known, but before now a realistic expression of dissipation has not been used in this variational method. Our new result (5.7) should give both a qualitative and quantitative description of a soliton in a homogenous quasi-one-dimensional warm condensate. This could be compared either experimentally or with simulations (see section 6.2.1).

5.1.3 Soliton Lifetime

Knowing the velocity we can calculate the soliton lifetime within the condensate. This is an important quantity as the lifetime is an experimentally measurable quantity.

$$T_d = \int_{t_i}^{t_f} dt = \int_{v_i}^{v_f} dv \left(\frac{dt}{dv}\right),$$

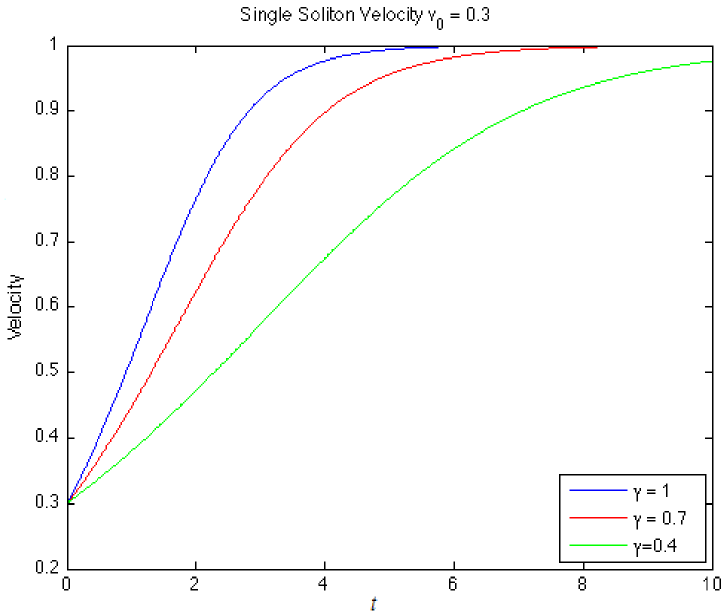

Figure 5.1: The velocity of a single soliton in a homogenous quasi-one-dimensional warm Bose-Einstein condensate. The dissipation rate γ is temperature dependent. The results show acceleration from initial velocity $v_0 = 0.3$ asymptotically approaching the speed of sound in the condensate.

with v given by (5.7). Solving this we get the single soliton lifetime under the effects of dissipation

$$T_d = \frac{3}{4\gamma}\log\left(\frac{v_f^2}{1-v_f^2}\frac{1-v_i^2}{v_i^2}\right). \tag{5.8}$$

To calculate the lifetime of a soliton with initial velocity v_i we would set $v_f = c$ ($c = 1$ in our scaling). As the velocity approaches c asymptotically, we find that the time diverges. However with any experimental measurement, especially under the effect of noise, there will be a speed at which the soliton is too shallow to be resolved. At this point we say the soliton has decayed. So letting $v_f = 1 - \varepsilon$, where ε is related to the resolution of the measurements, we have the lifetime

$$T_d = \frac{3}{4\gamma}\left[\log\left(\frac{1}{2}\left(\frac{1-v_i^2}{v_i^2}\right)\right) - \log\varepsilon - \frac{3\varepsilon}{2}\right] + \mathcal{O}\left(\varepsilon^2\right).$$

Changing the resolution ε, means changing the lifetime by some constant offset. We can write the lifetime in the form

$$T_d = \frac{3}{4\gamma}\log\left(\frac{(1-\varepsilon)^2}{2\varepsilon-\varepsilon^2}\right) + \frac{3}{4\gamma}\log\left(\frac{1-v_i^2}{v_i^2}\right).$$

In this form we see clearly that changing the resolution does not change the shape of the curve, but merely shifts it higher, leading to longer lifetimes. The dependence of soliton lifetime on initial velocity can be seen in figure 5.2.

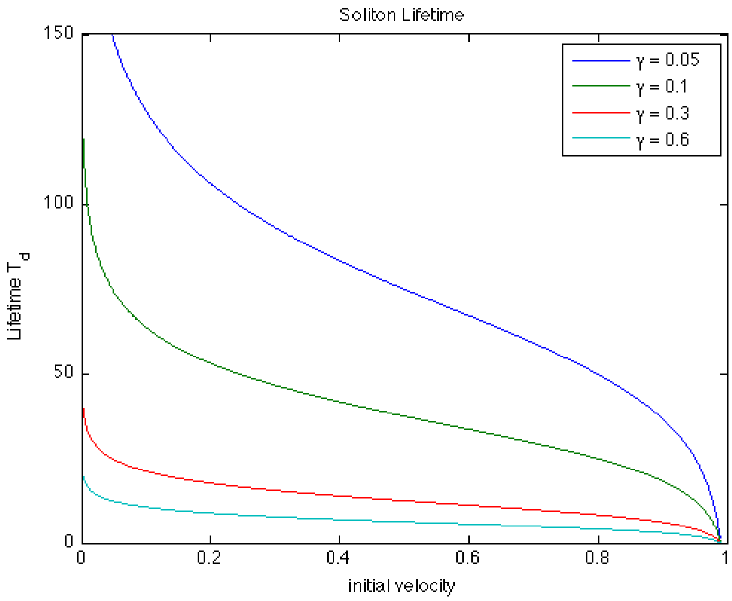

Figure 5.2: The lifetime of a soliton under the effects of dissipation in an infinite homogenous condensate. Here we assume that the soliton can be resolved up to a velocity $v_f = 0.99c$. Increasing the resolution velocity results in a constant offset increasing the lifetime of all states by the same amount.

For completeness we calculate the energy and momentum of a damped soliton in a homogenous infinite condensate using equations (4.16) and (4.17) respectively,

$$E_s = \frac{4}{3}(1 - v^2)^{3/2}, \tag{5.9}$$

$$I_s = 2v\sqrt{1 - v^2}, \tag{5.10}$$

where once again v is given by (5.7).

The velocity, momentum and energy are plotted as a function of time in figure 5.3. We see that the energy decreases with time as the dark soliton decays. The momentum initially increases as the soliton accelerates; but the decay of amplitude corresponds to a decrease in effective mass and soon this decreasing effective mass dominates the increasing velocity and the momentum reduces. This balancing effect means that we get the sensible results that a stationary soliton has no momentum and when the depth $B \to 0$, the momentum $I_s \to 0$.

5.1.4 Single soliton under the effects of noise

We would like to treat the effects of stochastic noise under the perturbation formalism. The stochastic noise has its origins in quantum fluctuations and is temperature dependent. As noted before it is also dependent on the dissipation in the system in such a way as to cause the system to approach the correct equilibrium state. Having derived a noise term $\xi(x,t)$, and determined the correct correlation strength according to the fluctuation

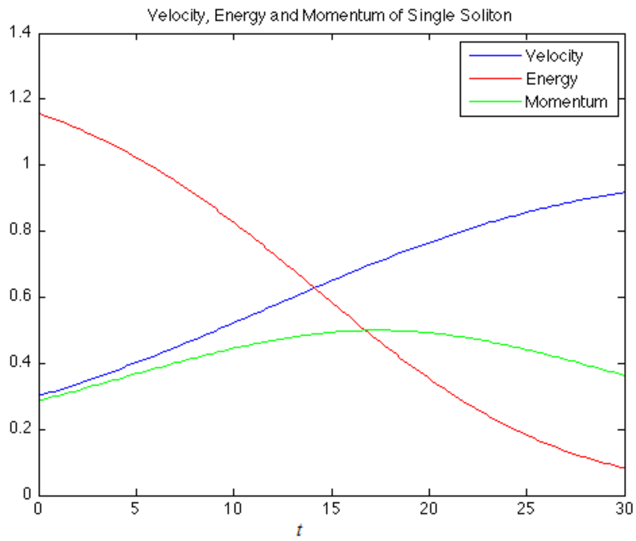

Figure 5.3: The velocity, momentum and energy as a function of time for a single dark soliton in a homogenous quasi-one-dimensional warm Bose-Einstein condensate. Dissipation causes the energy to decrease, the velocity to increase and the momentum to initially increase then decrease as the effective mass decreases faster than the increase in velocity.

dissipation theorem, we would naively expect that we could just substitute this expression into the right hand of the perturbation equations of motion (4.21). Doing this gives

$$\epsilon\Re\left(\int_{-\infty}^{\infty} R^*(u)\frac{\partial u}{\partial a_j}\right) = \epsilon\Re\left(\int_{-\infty}^{\infty} dx\xi^*(x,t)\frac{\partial u}{\partial a_j}\right).$$

The quantity of interest of stochastic functions is the non-vanishing correlation given by $\langle\xi^*(x,t)\xi(x',t')\rangle$. To calculate the correlation of this stochastic perturbation we consider

$$2\epsilon\Re\left(\int R^*(u)\frac{\partial u}{\partial a_j}\right) = dW_{a_j}(x,t) = \epsilon\left(\int dx\xi^*(x,t)\frac{\partial u}{\partial a_j} + \int dx\xi(x,t)\frac{\partial u^*}{\partial a_j}\right).$$

The correlation of this stochastic term is

$$\begin{aligned}\langle dW^*_{a_j}(x,t)dW_{a_j}(x',t')\rangle =&\epsilon\int dx\int dx'\langle\xi^*(x,t)\xi(x',t')\rangle\frac{\partial u(x,t)}{\partial a_j}\frac{\partial u^*(x',t')}{\partial a_j}\\&+\epsilon\int dx\int dx'\langle\xi(x,t)\xi^*(x',t')\rangle\frac{\partial u^*(x,t)}{\partial a_j}\frac{\partial u(x',t')}{\partial a_j}\\&+\mathbf{c.c},\end{aligned}$$

using the fact that $\langle\xi(x,t)\xi(x',t')\rangle = \langle\xi^*(x,t)\xi^*(x',t')\rangle = 0$. We obtain an expression for the correlation of the stochastic noise as a perturbation

$$\langle dW^*_{a_j}(x,t)dW_{a_j}(x',t')\rangle = \epsilon T\gamma\delta(t-t')\int_{-\infty}^{\infty} dx\left|\frac{\partial u}{\partial a_j}\right|^2. \tag{5.11}$$

In order to determine the strength of the stochastic noise as a perturbation on the Euler-Lagrange equations we must evaluate the integral in the above equation. Consider the equation corresponding to $a_j = B$, for our single soliton ansatz equation (5.1),

$$\frac{\partial u}{\partial B} = \tanh[D(x-x_0)] + i\frac{dA}{dB} = \tanh[D(x-x_0)] - i\frac{B}{A}.$$

$$\left|\frac{\partial u}{\partial B}\right|^2 = \tanh^2[D(x-x_0)] + \frac{B^2}{A^2} = \frac{1}{A^2} - \operatorname{sech}^2[D(x-x_0)].$$

The second term gives a finite result when integrated over all of space but the constant term diverges. This is quite a disturbing result. It seems that that the finite fluctuations (which mathematically could be made as small as you like by reducing ϵ), lead to some kind of infinite effect when treated as a perturbation.

The perturbed Euler-Lagrange equations were presented in [47]. In this paper the result appears to require only that the perturbation itself is small, in this case the correlation of the noise itself must be small; however for the origins of this equation one is referred to [34]. In this paper they use renormalization to calculate the effects of the soliton by subtracting the constant background. In particular when dealing with a perturbation, we want to calculate the effects of the perturbation on the soliton and not the effects on the background. The derivation of the perturbed Euler-Lagrange equations is carried out for a perturbation which vanishes as $|x| \to \infty$, and therefore does not have an affect on the background. The stochastic noise function $\xi(x,t)$ certainly makes a contribution for all x. The effects on the soliton can be found by subtracting the effects of the noise on the background. When we make this realization it is not surprising that the noise perturbation leads to an infinite effect. The soliton is on an infinite constant background and the perturbation over an infinite span will need to produce some infinite effect.

It is natural at this point to wonder if the damping calculations we did above are still valid, we must check that the effects of the perturbation do vanish as $|x| \to \infty$. We calculated the effects of dissipation as a perturbation of the form

$$\epsilon R(u) = -i\gamma\left(-\frac{1}{2}\frac{\partial^2 u}{\partial x^2} + (|u|^2 - 1)u\right).$$

Substituting our single soliton ansatz we find that

$$\begin{aligned}\epsilon R(u) = &-4\gamma AB^2\frac{\cosh^2[D(x-x_0)]}{(1+\cosh[2D(x-x_0)])^2} \\ &+ i4\gamma BA^2\frac{\cosh^2[D(x-x_0)]\sinh[D(x-x_0)]}{(1+\cosh[2D(x-x_0)])^3}.\end{aligned}$$

We can see from a plot of the functional forms of both the real and imaginary parts (figure 5.4), that the perturbation goes to zero as $|x| \to \infty$. The perturbation has an effect on only a small spatial segment and hence does not really affect the infinitely large background. We are now assured that our dissipative perturbation results are valid.

We were unable to separate the affects of the noise on the infinite background from that of the noise on the soliton. For a more complete discussion on our attempts to include noise we refer the reader to section 7.

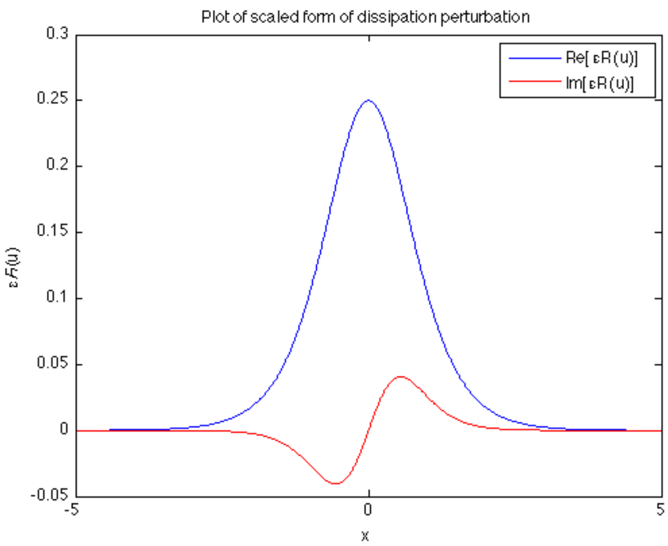

Figure 5.4: Plot of the functional form of dissipation for the single soliton in a homogenous Bose-Einstein condensate. The real part has the form $\cosh(x)^2/(1+\cosh(2x))^2$ and the imaginary part has the form $\cosh(x)^2\sinh(x)/(1+\cosh(2x))^3$. The effect of the perturbation goes to zero as $|x| \to \infty$.

5.2 Two interacting solitons

5.2.1 Two soliton interaction

The case of multiple soliton interactions is very interesting, for a rather in depth discussion including both analytic and experimental results we refer the reader to [44], this includes a description of the two interacting solitons discussed here analyzed using Bogoliubov-de Gennes equations. The case of two counter propagating solitons can be considered under the Lagrangian perturbation formalism. In [47], Kivshar and Królikowski calculate the Lagrangian for two weakly interacting solitons. The ansatz used is that of the product of two identical solitons travelling in opposite directions,

$$u = (B\tanh z_+ - iA)(B\tanh z_- + iA), \tag{5.12a}$$
$$z_\pm = B(x \pm x_0). \tag{5.12b}$$

The relative separation between the two solitons is $2x_0$. This is a very symmetric form of two dark solitons on a common background which just slightly overlap, and therefore are only weakly interacting. In the calculation of the Lagrangian the approximation that the two solitons are widely separated was made, that is the separation of the solitons is much larger than the soliton width. Formally we assume $x_0 \gg B^{-1}$. Substituting (5.12) into the Lagrangian (4.18) Kivshar and Królikowski found,

$$L = 2L_0 + 4\frac{dB}{dt}[AB\tanh(2x_0B)]^{-1} + 16B^5\exp(-4x_0B), \tag{5.13}$$

where

$$L_0 = 2\frac{dx_0}{dt}\left[-AB + \tan^{-1}\left(\frac{B}{A}\right)\right] - \frac{4}{3}B^3.$$

L_0 is the Lagrangian of the isolated soliton (5.2), with $B = D$. The last exponential term in (5.13) accounts for the interaction of the solitons. The second term can be neglected when the separation $2x_0$ is large. So the effective Lagrangian is

$$L' = 4\frac{dx_0}{dt}\left[-AB + \tan^{-1}\left(\frac{B}{A}\right)\right] - \frac{8}{3}B^3 + 16B^5\exp(-4x_0B).$$

Using the Euler-Lagrange equations we get,

$$-64B^6\exp(-4x_0B) + 8B\frac{dA}{dt} = 0 \tag{5.14}$$

$$8\frac{dx_0}{dt}\frac{B^2}{A} - 8B^2 - 16B^4\exp(-4x_0B)\,(5 + 4x_0B) = 0. \tag{5.15}$$

Remembering that $x_0 \gg B^{-1}$, so that $(5 + 4x_0B) \approx 4x_0B$, Kivshar and Królikowski found

$$\frac{dA}{dt} = 8B^5\exp(-4x_0B) \tag{5.16a}$$

$$\frac{dx_0}{dt} = A\left[1 + 8x_0B^3\exp(-4x_0B)\right]. \tag{5.16b}$$

Calculating $\frac{d^2x_0}{dt^2}$,

$$\begin{aligned}\frac{d^2x_0}{dt^2} =& 8B^5\exp(-4x_0B) + 64Bx_0\exp(-8x_0B)\\ & A^2(8B^3\exp(-4x_0B) - 32x_0B^4\exp(-4x_0B) + 128\exp(-8x_0B)).\end{aligned}$$

Assuming weakly interacting almost dark solitons ($A^2 \ll 1$) and use the fact that $x_0B \gg 1$ implies that $\exp(-4x_0B) \gg \exp(-8x_0B)$,

$$\frac{d^2x_0}{dt^2} = 8B^5\exp(-4x_0B).$$

So

$$\frac{d^2x_0}{dt^2} = -\frac{dV(x_0)}{dx_0}, \quad V(x_0) = 2B^4\exp(-4x_0B). \tag{5.17}$$

Thus the soliton paths are described by a classical particle in a potential. This is a repulsive potential so the solitons repel each other. The strength of the repulsion depends on the soliton depth B^2, and indicates that 'blacker' solitons interact more strongly than 'lighter' solitons. In fact in the limit that $B^2 \to 0$, we see from (5.15), $\frac{dx_0}{dt} \to A = v$.

Given the number of approximations made one might question the validity of this result. In fact for the case of two solitons interacting with no potential, an exact solution can be found through inverse scattering theory and confirmed by direct substitution. Using the results from [3], and assuming the solitons are well separated Kivshar and Królikowski produced the result,

$$\frac{d^2x_0}{dt^2} = \frac{B^3\tanh(2x_0B)}{\cosh^2(2x_0B)} \tag{5.18}$$

for the soliton minimum intensity. The potential associated with this repulsive potential is

$$V(x_0) = \frac{1}{2}\frac{B^2}{\cosh^2(2x_0B)}.$$

For high contrast solitons with a large spatial separation this potential has the same asymptotic behaviour as that obtained with the help of the Lagrangian formalism (5.18).

The exact solution for the collision of two dark solitons is given in [3] after much simplification as,

$$\Psi = \frac{(2a_3 - 4a_1)\cosh(\mu t) - 2\sqrt{a_1a_3}\cosh(2px) + i\mu\sinh(\mu t)}{2\sqrt{a_3}\cosh(\mu t) + 2\sqrt{a_1}\cosh(2px)}e^{i2a_3t},$$

where a_1, a_2, a_3, p, μ are all parameters which have been determined. We see that even in the simple case of two solitons colliding, the exact answer is difficult to work with analytically and hard to deal with the dependence of its parameters intuitively. The Lagrangian formalism on the other hand gives a solution which is easy to grasp, the product of two single solitons, and gives equations of motion for the parameters. This is an example of the power and clarity gained from using a Lagrangian approach. Furthermore the case of two solitons under a non-conservative force such as dissipation, gives a non-integrable NLS equation and does not have an exact solution. The Lagrangian approach will yield an answer using the perturbation techniques discussed above.

5.2.2 Dissipation in the two soliton system

Building on the previous section and Kishar & Królikowski's results, we can calculate the effects of dissipation on the dynamics of two weakly overlapping solitons. Assuming the same functional form for u, namely (5.12) and (5.12b), we would carrying out the same process as for the undamped case arrive at the same Euler Lagrange equations (5.14) and (5.15). However instead of solving

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0,$$

we are solving

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 2\epsilon\Re\left(\int_{-\infty}^{\infty} dx R^*(u)\frac{\partial u}{\partial q_i}\right),$$

with

$$\epsilon R(u) = -i\gamma\left(-\frac{1}{2}\frac{\partial^2 u}{\partial x^2} + (|u|^2 - 1)\right)$$

the same dissipation term as for the single soliton. The difficulty comes in solving these perturbations under the same approximation scheme as the Euler-Lagrange equations were derived; that is assuming that $x_0 \gg B^{-1}$. This leads to a number of approximations, but the most relevant being the vanishing of the integral over space of terms of the form

$$\mathrm{sech}^{2n}(z_-)\mathrm{sech}^{2m}(z_+) = \mathrm{sech}^{2n}(z_-)\mathrm{sech}^{2m}(z_- + 2x_0B) \approx 0, \quad n, m \in \mathbb{N}$$

when $2x_oB \gg 1$. The $\mathrm{sech}^{2n}(x)$ function is a sharply spiked function which quickly decays away to zero. If the separation of the two spikes is much larger than their width, the resulting product is essentially zero, as illustrated in figure 5.5 for $n = m = 1$.

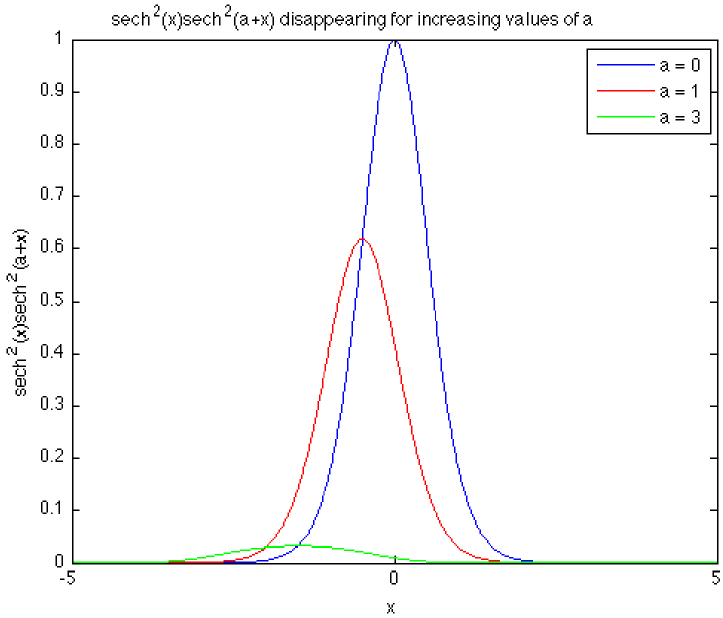

Figure 5.5: Plot of $\mathrm{sech}^2(x)\mathrm{sech}^2(a+x)$ for increasing separation a. As the separation increases the values and area of this function go to zero. This means that these terms produce a negligible result and can be dropped. The result for $\mathrm{sech}^{2n}(x)\mathrm{sech}^{2m}(a+x)$ with integer values of n, m is similar.

The calculations are formidable but the results are rather simple:

$$\epsilon\Re\left(\int_{-\infty}^{\infty} dx R^*(u)\frac{\partial u}{\partial x_0}\right) = \frac{8}{3}\gamma A^3 B^3,$$

and

$$\epsilon\Re\left(\int_{-\infty}^{\infty} dx R^*(u)\frac{\partial u}{\partial B}\right) = \gamma AB\coth(2x_0 B) + \frac{8}{3}\gamma A^3 B^2 x_0.$$

This gives us the following equations,

$$8\frac{dx_0}{dt}\frac{B^2}{A} - 8B^2 - 16B^4\exp(-4x_0B)(5+4x_0B) = 2\gamma AB\coth(2x_0B) + \frac{16}{3}\gamma A^3B^2x_0,$$

and

$$-64B^6\exp(-4x_0B) + 8B\frac{dA}{dt} = \frac{16}{3}\gamma A^3B^3.$$

These give the following equations of motion

$$\frac{dA}{dt} = 8B^5\exp(-4x_0B) + \frac{2}{3}\gamma A^3B^2, \tag{5.19a}$$

$$\frac{dx_0}{dt} = A[1+8x_0B^3\exp(-4x_0B)] + \gamma\left(\frac{1}{4}\frac{A^2}{B} + \frac{1}{2}\frac{A^2}{B}\exp(-4x_0B)) + \frac{2}{3}A^4x_0\right). \tag{5.19b}$$

This is a new result. Kivshar and Królikowski's result (5.16), gives a description of two widely separated solitons in a condensate described by the GPE. The new dissipative result for two solitons describes the two solitons according to the dissipative GPE. The explicit stochastic term has been neglected due to the difficulties of the non-zero background. However, we would expect that our result for dissipation would be related to the averaged dynamics over an ensemble of systems (for more information on this relationship see section 7). From these equations we can describe the position of the soliton (the position of the minima), and the velocity. Of course, from the soliton velocity we can determine the width.

If we take the weakly interacting, almost black solitons ($A^2 \ll 1$), then the soliton coordinate x_0 behaves as if it is in the same potential (5.17) as the system with no dissipation. This result is not surprising when we remember that the velocity is connected to the 'blackness' of the soliton. Very black solitons have very low velocities, $\frac{dx_0}{dt} = A[1 + 8x_0 B^3 \exp(-4x_0 B)]$, and hence dissipation would have a minimal affect. When A is large ($B \ll 1$) and hence velocity is large, we find that dissipation will have a larger effect on on the velocity. The first and second term of the dissipation contribution are proportional to A^2, with the second term being position dependent (as is the third term).

Chapter 6

Numerical Results

There are many interesting phenomena to consider via a numeric simulation of dark solitons under the effect of damping and noise. Using code which implements the simple growth SGPE (3.32), we will make a direct comparison between the numeric simulation and our variational results for a damped soliton. In addition we will look at the effects of noise on the dynamics of solitons. The introduction of noise will introduce a number of interesting phenomena and highlight the importance of providing some analytic scheme in which to treat dark solitons under the effects of noise.

6.1 SGPE in Fourier Basis

Having obtained approximate analytic solutions to describe the effects of damping on a single soliton and two widely separated solitons in a homogenous quasi-one-dimensional warm BEC; we wish to compare the results to numeric simulations of the simple growth SGPE (3.32) formalism. The code I used was developed by Blair Blakie and is a one-dimensional implementation of the SGPE. The code solves the differential equation via a Fourier transform (The full details of the code are unimportant to the project). For numerics the Fourier transform is a discrete Fourier series and the wave function is periodic. In order to get smooth results the wave function must match at the endpoints. The code is thus best suited to periodic systems. This periodic boundary condition was dealt with in two ways, allowing us consider two systems. In the first case we used an external potential to create a finite homogenous condensate. In the second case we considered an explicitily periodic system and identified this with a torus.

6.1.1 Solitons in a Torus

We considered an explicitly periodic system, and identified this with a torus by identifying the end points as a single point. We simulated two dark solitons for $-L/2 \leq x \leq L/2$ with the identification of the end points as a single point. This is topologically a circle. This situation could be experimentally realized as two solitons in a tightly confined (cross section radius) torus. For the homogenous case, $V^{\text{ext}}(x) = 0$, the code necessitates the use of two identical counter propagating solitons. As seen in section 2, the phase shift across a single soliton is non-zero. A single soliton cannot exist in a torus due to this phase shift. Two counter propagating solitons of the same amplitude (and hence same velocity and phase shift) will produce a continuous phase and match at the endpoints allowing them to exist in a torus. The study of solitons in a torus is not just interesting from a theoretical point of view but has been studied experimentally [13].

6.1.2 Finite Homogenous Condensate

By introducing a potential we can create a homogenous condensate over some finite distance, which will hold a single soliton, and allow us to test our single soliton damping solution (5.7). Although all analytic results are derived for an infinite homogenous system, we can increase the size of our system (or equivalently narrow our healing length) such that the finite condensate is much larger than the soliton width, giving us a system with properties approaching those of an infinite system.

We create this semi-infinite homogenous condensate by introducing a potential at the ends which will confine the condensate spatially and leave us with a homogenous condensate within some finite region. We keep $V^{\text{ext}}(x) = 0$ for $-3L/8 < x < 3L/8$ and raise the potential to $A \gg \mu$ ($A = 5\mu$) smoothly outside this region. This is implemented via $A\cos^2(x)$ for $-L/2 \leq x \leq -3L/8$ and $A\cos^2(x + \pi/2)$ for $3L/8 \leq x \leq L/2$. By setting L large enough we can approximate a single soliton in a homogenous condensate for $-3L/8 < x < 3L/8$. The fact the condensate decays to zero at $\pm L/2$ means the continuous boundary conditions will be satisfied trivially for any imprinted wavefunction, avoiding the phase jump problem. To find the eigenstate for the background condensate in this case we propagate the homogenous state through time with $\gamma = 1$. As time progresses the excitations decay away leaving only the time independent steady state, which is then the eigenstate of the system with the new potential. When the system reaches a steady state we imprint our required solitons onto this state and use it as the initial condition (see figure 6.1).

6.2 Single Soliton in semi-infinite homogenous BEC

6.2.1 Single Soliton Damping

Now we are able to model a single soliton in a homogenous condensate, we can test our dissipative result (5.7). The velocity is dependent on both time and the damping coefficient γ. In order to make a quantitative comparsion we must use the correct scaling. We can scale the simulation in units of healing length ξ, the same as in section 4.4.1. This gives us a simple interpretation of length in our plots.

$$\bar{x} = x_0 x,\ x_0 = \xi, \quad \bar{t} = t_0 t,\ t_0 = \frac{\xi}{c} \quad \mu = g_{1D} n_0 \quad c = \sqrt{\frac{\mu}{m}}.$$

The healing length is given by

$$\xi = \frac{\hbar}{\sqrt{m\mu}} \ \Rightarrow, x_0 = \frac{\hbar}{\sqrt{m\mu}}\ t_0 = \frac{\hbar}{\mu},$$

which corresponds to the dimensionless scaling. We again make the identification

$$\bar{\gamma} = \frac{\hbar}{k_B T}\gamma,$$

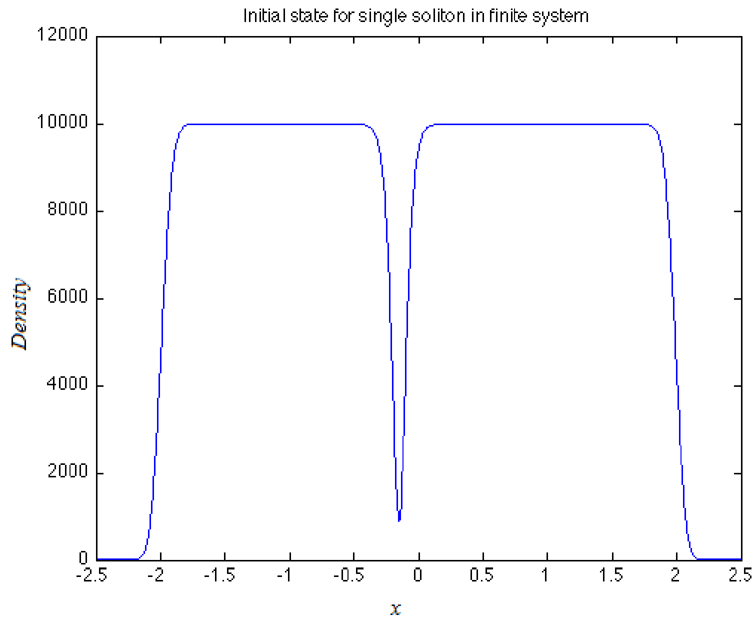

Figure 6.1: Plot showing a single soliton imprinted on a groundstate of the GPE. The external potential is zero within the condensate, and provides confinement near $x = \pm L/2$. The eigenstate is found by time propagation with $\gamma = 1$. The fact that the condensate is zero at $x = \pm L/2$ ($L = 5$) means that the continuous boundary conditions $\psi(L/2) = \psi(-L/2)$ are satisfied trivially.

and in this healing length scaling we can directly compare the numerical results to the analytically derived results.

We can now make a direct comparison of our analytic Lagrangian perturbation damping result (5.7) with a numerical simulation of dark soliton decay described by the damped GPE. We can see the results of the simulation and our analytic result for an initial velocity $v_0/c = 0.3$ in figure 6.2. The results for other initial velocities are very similar. Clearly there is rather good quantitative agreement between the two velocity curves. There is some deviation between the numerical results and the analytic prediction; however the analytic prediction could be improved upon to gain increased accuracy if required. Any increase in accuracy comes at a cost of increased mathematical difficulties. From the simulation there is some initial "shedding" where the soliton is radiating energy. A more complete ansatz would include this effect. Also the simple perturbative treatment could be improved using a more complete description such as Collective Variable theory [43] for example. For more details on improvements see section 7.

6.3 Noise Induced dynamics

6.3.1 Single Soliton Noise Induced Motion

If we consider the equations of motion for a single soliton with initial velocity $v(0) = 0$, we find that the soliton will remain at rest $v(t) = 0$. A black soliton has zero velocity. In this case the stationary solution which is an eigenstate of the GPE is also an eigenstate of the

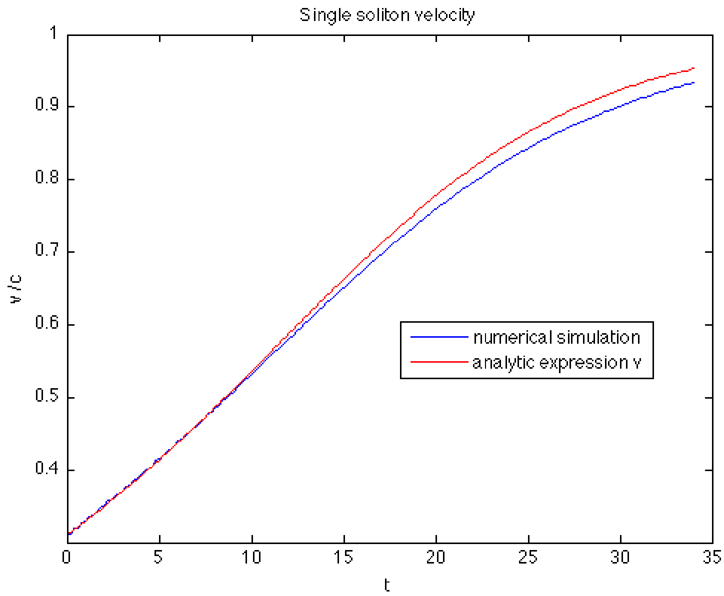

Figure 6.2: Plot showing the velocity of a decaying soliton. There is a strong correlation between the numeric results and the Lagrangian analytic expression (5.7). The initial velocity is $v = 0.3c$ and $\gamma = 0.1$.

dissipative GPE (SGPE without the stochastic noise). The reason for this is the fact that there is no symmetry breaking; a soliton has to decay by moving in a specific direction. We find that when noise is included, the black soliton is no longer an eigenstate. The inclusion of noise coupled with the fact that damping causes acceleration, means that an initially stationary soliton can in fact gain momentum through the fluctuations and undergo damping induced acceleration, see figure 6.3. For particles with a positive effective mass, any small momentum gained through fluctuations would simply decay away through dissipation. For solitons the stochastic fluctuations cause the density and hence velocity to fluctuate. When the fluctuations are large enough or add constructively for some short time period the soliton has gained enough momentum that it will accelerate and the affects of the fluctuations will not be large enough to stop the runaway soliton. Thus, stochastic noise will induce soliton acceleration and decay, see figure 6.3. The stationary soliton eigenstate of both the GPE and damped GPE is no longer an eigenstate with the introduction of noise. The soliton may gain momentum through the noise induced decay, but this is only temporary. As mentioned in section 5.1.2 the soliton initially increases its momentum as the velocity increases but ultimately the momentum goes to zero as the velocity approaches the speed of sound.

6.3.2 Dissipative GPE vs averaged SGPE

It is interesting to consider the relationship between the dissipative GPE and the averaged results of the SGPE. In our computational units we have the one-dimensional SPGPE,

$$\mathrm{d}\bar{\psi}(\bar{x},\bar{t}) = -i\mathcal{P}[\bar{L}\bar{\psi}\mathrm{d}\bar{t}] + \mathcal{P}[\bar{\gamma}(1-\bar{L})\bar{\psi}\mathrm{d}t] + d\bar{W}(\bar{x},\bar{t}), \tag{6.1}$$

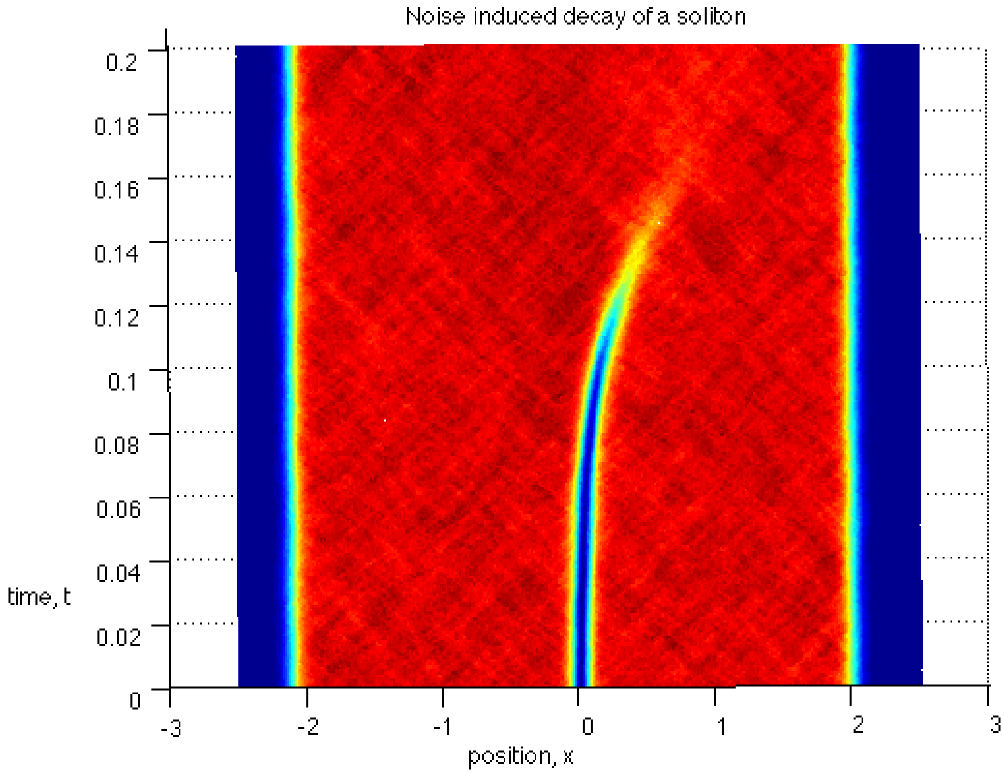

Figure 6.3: Noise induced acceleration and decay of an initially stationary soliton. A stationary soliton is an eigenstate of the GPE and damped GPE but not the SGPE. The fluctuations in velocity will cause runaway acceleration and decay of the soliton.

where

$$\bar{L} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \bar{g}_{1D}|\bar{\psi}|^2, \quad \bar{g}_{1D} = \frac{g_{1D}}{\hbar c},$$

is the Gross-Pitaevskii operator and

$$\langle d\bar{W}(\bar{x}, \bar{t})^* d\bar{W}(\bar{x}', \bar{t})\rangle = 2\bar{\gamma}\bar{T}.$$

We have $\bar{\gamma} = \frac{\hbar}{K_B T}\gamma$ and $\bar{T} = \frac{k_B T}{\mu}$ as in section 4.4.1.

There has not been a lot of research done on the dynamics of solitons under the effects of stochastic noise. Much work has been done to model finite temperatures using phenomenological [6], quasi-particle scattering [22] and quantum effects [21]. These predict increasing amplitude solutions "anti-damping", but not of the form which gives quantitative agreement to simulations of the DGPE as our result does. Experiments show that there is significant variation in soliton lifetimes. One of the most recent studies done on stochastic effects is by Cockburn *et al.* [17]. In this study it is found that the averaged stochastic results correspond quite well with the dissipative GPE. It is certainly of interest to investigate whether the average dynamics is completely captured by the dissipative GPE or if the noise has some effect on dissipation.

We ran the simulation of the decay of a single soliton in a semi-infinite homogenous condensate using the full simple growth SGPE formalism 100 times, and compared the

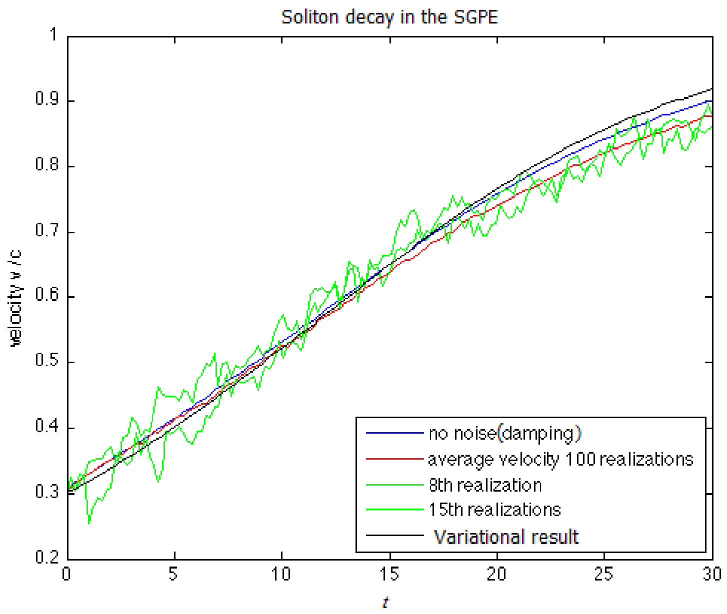

Figure 6.4: The effects of noise on the dissipation rate of a dark soliton in a large homogenous condensate. The stochastic noise reduces the effects of damping, seen clearly in figure 6.6. The temperature here is $T = 0.1\mu$

average result with that of the numerical simulation of the dissipative GPE which has no stochastic term. This was repeated at a number of temperatures (see figures 6.4-6.6).

The results are all for solitons with initial velocity $v_0 = 0.3c$, if the stochastic average was in fact equivalent to the dissipative GPE, then we would expect that the average decay velocity over 100 realizations would be converging to the dissipative GPE result. Our results show that there is convergence but that the decay rate seems to be decreased. The size of the discrepancy is dependent on temperature, with the average stochastic results approaching the dissipative GPE in the limit of $\bar{T} \to 0$. This is clearly a very interesting result and is contrary to vortex decay behaviour found by Rooney *et al.* [39]. The damping is affected by noise with the fluctuations causing the damping to be decreased on average. The reason for this is far from clear although we do make a suggestion in the discussion. Clearly the results highlight the need for an analytic description that includes the effect of stochastic noise.

6.4 Two Solitons in a Torus

6.4.1 Soliton Decoupling in a Toroidal System

Coupled Solitons in a torus

Two solitons in a torus described by the GPE are relatively uninteresting. The solitons must be identical and counter-propagating in order to have a periodic wavefunction. Under time evolution the solitons simply propagate around the torus and pass through each other. When damping is introduced, the two solitons simply accelerate around the

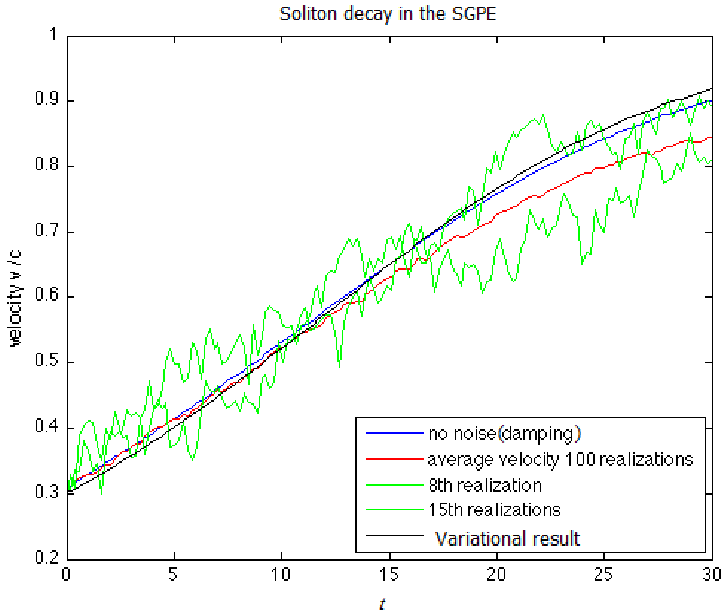

Figure 6.5: The effects of noise on the dissipation rate of a dark soliton in a large homogenous condensate. The stochastic noise reduces the effects of damping, seen clearly in figure 6.6. The temperature here is $T = 0.3\mu$.

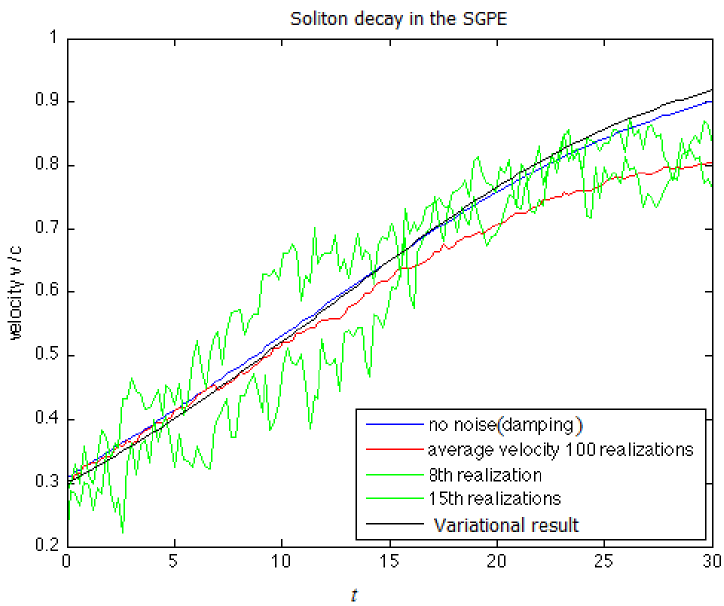

Figure 6.6: The effects of noise on the dissipation rate of a dark soliton in a semi-infinite homogenous condensate. The stocahstic noise reduces the effects of damping. The temperature here is $T = 0.6\mu$.

torus passing through each other until they decay away. The two solitons do interact with each other (see section 6.4.2) but must remain identical and counter-propagating. There is a certain irreducible symmetry in this setup until we introduce thermal noise.

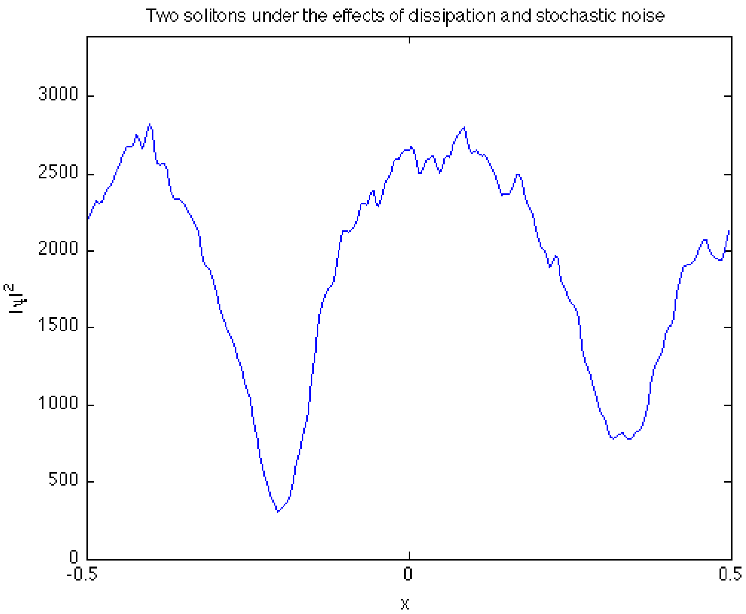

Figure 6.7: Plot showing the time evolved state of two initially equal oppositely propagating solitons in homogenous quasi-one-dimensional torus under the effects of dissipation and noise. The initial pure solitons are required to have the same depth in order to provide a continuous phase around the torus. The effects of fluctuations mean the phase is no longer smooth but is continuous without requiring equal depth solitons.

Noise Decoupling

Stochastic noise causes fluctuations in the condensate and the amplitude of the solitons will also fluctuate. The fluctuations in the amplitude cause the velocity and phase shift of the soliton to fluctuate also. Two initially identical counter-propagating solitons are no longer identical after time evolution under the SGPE, as shown in figure 6.7. The stochastic noise causes fluctuations in the phase so that we no longer have a smooth definite valued phase shift from each of the solitons. The fluctuations in phase mean we can satisfy the condition of continuous phase around the torus without identical solitons. Without stochastic fluctuations the two solitons are coupled, they must have the same velocity and phase jump. The geometry of the toroidal system causes great restrictions on the available soliton configurations for both the non dissipative GPE and the dissipative GPE. The introduction of noise breaks this coupling and symmetry. The size of the fluctuations determine how badly this symmetry can be broken. The wavefunction and phase must be continuous around the torus; the size of the phase discrepancy between the two solitons (and hence the depth and velocity) has an upper bound based on the strength of the fluctuations in order that the phase can match up. The coupling of these two counter propagating solitons is directly related to the strength of the stochastic noise. This is a very interesting phenomena and an analytical description of dark solitons under the influence of noise would most likely show some interesting relationship between the strength of the noise and the asymmetry of the solitons.

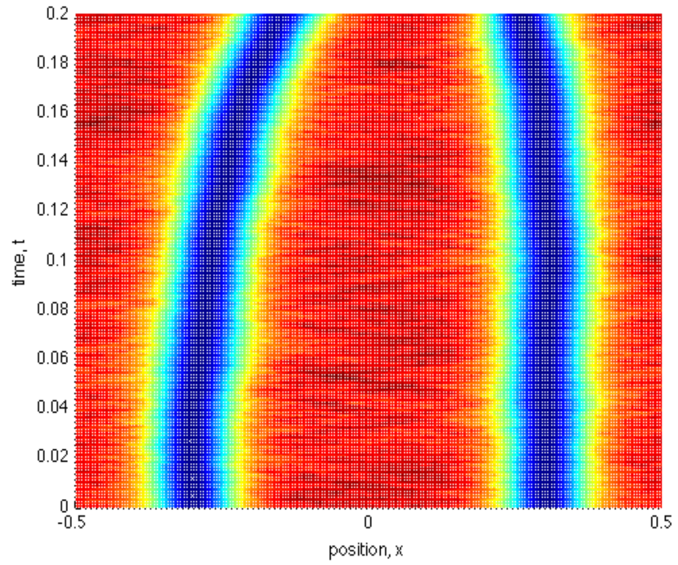

Figure 6.8: Noise induced acceleration of two initially stationary solitons. The existence of two stationary solitons is restricted to cases of exceptional symmetry where the soliton seperation is at a maximum.

6.4.2 Two Soliton Interactions and Stationary eigenstates

While we predicted that a single soliton with initial velocity of zero is an eigenstate of the Hamiltonian and will remain stationary, the same is not true of two solitons. As mentioned in section 5.1.2, solitons act as particles of negative effective mass. Thus a system consisting of two stationary solitons will experience a repulsive force. This force extends over all space but is very weak at large distances yielding an effective potential

$$V(x_0) = 2B^5 \exp(-4x_0 B).$$

This repulsive potential is always present which means that we cannot usually have two stationary solitons. However Figure 6.8, does show two initially stationary solitons. This is a peculiarity of the toroidal system. If the separation of the two solitons is $L/2$ then the two solitons are already at a maximum distance from each other. It is only when we have such symmetry with the solitons at maximum displacement that a two soliton stationary eigenstate can exist.

Chapter 7

Discussion

7.1 A Variety of Variational Methods

In our attempts to provide some analytic technique to describe noise in the infinite homogenous system, we looked at a number of other variational techniques. The two main techniques considered were collective variable theory and a variational path integral method. These methods are more complete than the Lagrangian perturbation method, and must be used if we wish to consider the case when damping and noise are not small effects. It was hoped they would provide a solution to the problem of incorporating noise into the variational method. Unfortunately they did not provide an immediate answer.

The collective variable theory for optical solitons in fibers is rigorously described in a paper by Dinda *et al.* [43]. Collective variable theory is very similar to the Lagrangian variational method considered in the project. In order to simplify the description of pulse dynamics we associate new variables (collective variables (CVs)) with local phenomena such as amplitude or position of minima etc. One must find a transformation which allows the original field equations to be described completely in terms of the CV's, and equations of motion are derived for the CV's. There are a number of methods to obtain these equations of motion. Dinda *et al.* [43] uses a projection-operator formalism to treat the NLS equation for optical solitons in fibers, the case of dark solitons in a condensate is similar. Dinda outlines the general procedure for applying the projector-operator formalism to find the equations of motion of CVs; describing not only the soliton motion but the residual field, which includes the soliton dressing and any radiation coupled to the soliton motion. An efficient numerical procedure for solving the collective variable method is also considered. The method has even been used to derive the exact equations of motion for the center of mass of a discrete sine-Gordon kink and the frequency of its small oscillations in the Peierls-Nabarro well [10]. While not directly applicable to dark solitons in a condensate, this shows that the technique is quite powerful.

The true strength of the collective variable technique is that it avoids the use of an explicit Lagrangian. While there is a renormalized Lagrangian for the NLS equation (4.18), a Lagrangian does not exist for the full SGPE description. Without a Lagrangian we had to approximate the deviation from the NLS equation as a perturbation. This limited us to small perturbations that vanish asymptotically in space. In fact the perturbed Lagrangian approach corresponds to the lowest-order or 'bare approximation' of the projection operator approach. For details on how the collective variable method produces variational results bypassing the use of a Lagrangian we refer the reader to [1]. The Lagrangian variational method has been used to treat stochastic noise in a limited way [2, 8, 37], see section 7.2.

The other method considered was a variational path integral method. Duine and Stoof [20], use a variational path integral method to determine the stochastic dynamics of trapped Bose-Einstein condensate. The paper explicitly contains a procedure to include the effects of stochastic noise when considering a Gaussian ansatz and derives a path integral expression from the stochastic differential equation giving an effective action. Ultimately one wishes to solve the path integral

$$P[\psi, \psi^*; t] = \int_{\psi(\mathbf{x},t)=\psi(\mathbf{x})}^{\psi^*(\mathbf{x},t)=\psi^*(\mathbf{x})} \mathcal{D}\psi^* \mathcal{D}\psi \exp\left\{ \frac{i}{\hbar} S^{\text{eff}}[\psi^*, \psi] \right\},$$

where S^{eff} is an effective action, $S^{\text{eff}} = \int_{t_0}^{t} \int d\mathbf{x} \mathcal{S}(\mathbf{x}, t)$. To solve this path integral using a variational approximation we substitute our ansatz into $\mathcal{S}$ and proceed as in the Lagrangian variational method considered in this project. This method gives a very good description of the stochastic dynamics of a Gaussian condensate under the effect of thermal noise. Given the renormalized Lagrangian associated with the SGPE, we could find the renormalized Hamiltonian (via the Legendre transform), and it should be possible to derive a renormalized effective action which could describe the dynamics of a dark soliton. However, it is far from assured that things are as simple as that.

The more complete descriptions come at the cost of more complex mathematical equations. Given the difficulty of solving the variational perturbation theory for two identical counter propagating solitons in a homogenous condensate, and the approximations made to produce an answer, complexity is a considerable issue. Although the collective variable theory and variational path integral method provide a more complete description of the dynamics of a pulse in a warm condensate they do not solve our problem concerning noise in the infinite system.

The fundamental problem when dealing with dark solitons in an infinite system is the background condensate. Dark solitons correspond to a dip in density and can clearly only exist within some nonzero background. In an infinite system we have an infinite background condensate. This means that the energy and momentum of the system is infinite. We already ran into this problem when determining the Lagrangian density of a dark soliton. The solution was to subtract the background and consider the energy difference between the soliton and the homogenous system as the energy of the soliton. When it comes to describing the affect of stochastic noise on the system we run into the same problems. The background is affected by the noise and this produces an infinite result. When considering a Gaussian system where the condensate density decays to zero as $|x| \to \infty$, we do not have this problem. The infinite background produces the problem, the collective variable theory and path integral method do not provide an immediate solution to this. In practice any system will be finite, so we know that the physical quantities are well defined and we simply require a means to mesh the calculational expedience of infinite systems with the practical reality of experiments.

7.2 A Noisy Discussion

The solution to the problem of infinite noise correlation is to subtract the effects of noise on the background from the total noise, leaving a finite contribution associated with the soliton. This is a general problem for perturbations which do not disappear as $|x| \to \infty$. There does not seem to be a well defined procedure to handle spatially dependent perturbations extending to infinity. Kivshar and Yang provide a solution to the problem of a background perturbation extending to spatial infinity [34]. To calculate the effects on the background one considers the limit $|x| \to \infty$, where the soliton is not disturbing the background, and consider the evolution of the nonpropagating background $u_b(t)$. When the perturbation converges to some spatially independent value as $|x| \to \infty$ we obtain the perturbed Lagrangian density,

$$i\frac{\partial u_b}{\partial t} - |u_b|^2 u_b = \epsilon R(u_b).$$

The second derivative term $\partial_{xx} u_b$ is now zero.

This simplification does not work when we are considering noise which is position dependent and does not converge to some spatially independent function in the limit $|x| \to \infty$. We cannot ignore the $\partial_{xx} u$ term and need a position dependent non-propagating background solution. This gives us the perturbed nonlinear Schrödinger equation (4.19), which we do not know how to solve for a continuous wave solution under the effects of stochastic perturbations. It is not really clear how to resolve the problem of stochastic fluctuations on an infinite system. While stochastic perturbations have been dealt with in the Lagrangian variational approach or the related collective variable approach [2, 8, 37], there has been no cases of both position and time dependent noise solved on an infinite background to the best of my knowledge. In [37] the case of multiplicative noise $\eta(r,t)\psi$ is dealt with using the collective variable method. However they describe a light soliton with the ansatz $\psi(r,t) = A(t)\mathrm{sech}[r/B(t)]\exp[i\alpha(t)r^2 + i\beta(t)]$, which decays away and means that the noise goes to zero as $r \to \infty$. This means they do not suffer the same problem as in the dark soliton case on an infinite background. Both papers [8] and [2] deal with position independent stochastic functions $\sigma(t)$. In [8] it is remarked that position dependent noise is a harder problem and refers the reader to [37]. The path integral method of Duine and Stoof [20] is solved for a Gaussian ansatz and once again avoids this problem.

The best prospect for handling the noise analytically appears to be to simply treat it as a perturbation of a finite system (such as a condensate confined in a trap) and this will be the focus of future work. Dark solitons on a background of finite extend have been looked at by Kivshar and Yang. The problem of a dark soliton on a finite width background is not strictly speaking a perturbative problem. However it can be reduced to a perturbative problem by first considering the background evolution. They consider the case of a background described by the NLS equation and derive a solution for the background which spreads reduces its intensity with time. Ultimately they find that a dark soliton solution on this background varying wave can be described as a perturbation. This example is not directly applicable for the SGPE as the background undergoes interactions with the non-condensate band in the form of scattering, see section 3.9.2 and section 3.9.3. However it does indicate that a finite system can (at least in some cases) be described by the Lagrangian perturbative formalism.

In section 6.3.2 we found the interesting result that the averaged effects of noise results are temperature dependent and result in a decreased damping rate. As mentioned before this result is contrary to vortex behaviour found by Rooney *et al.* [38]. The higher the temperature the larger the deviation. The reason for this is not clear at this point. Our suggestion is that damping induced acceleration, a special soliton property, is being cancelled to some extent by the more familiar damping deceleration. Thermal noise causes the wavefunction to fluctuate; at high temperatures $\bar{T}$, or high velocity the fluctuations are larger relative to the soliton depth. Under these conditions the wavefunction bears less of a resemblance to a soliton. In this case there might be dissipative deceleration component associated with this less well defined soliton profile. Needless to say this is an exciting result which warrants further investigation.

Chapter 8

Conclusions

We have considered dark soliton dynamics in a one dimensional Bose-Einstein condensate at high temperatures, where the effects of the thermal cloud are significant. Using the simple growth SGPE (3.32), we managed to find a renormalized perturbed Lagrangian (4.24) which includes the interactions of the thermal cloud and the condensate in the form of dissipation and stochastic noise. The establishment of this Lagrangian was central to our variational study of the dynamics of a Bose-Einstein condensate in an infinite system. Having determined the Lagrangian we could now try any number of solutions involving imprinting against a background. From this Lagrangian we were able to derive an expression for a damped soliton in an infinite homogenous condensate (5.7). This solution shows good quantitative agreement with numerical simulations of the damped GPE. We obtained expressions for the soliton lifetime as a function of initial velocity as well as the energy and momentum as a function of time. An expression for two damped counterpropagating solitons was also calculated (5.19). The Lagrangian variational solution for a single damped soliton in an infinite system shows good quantitative agreement with the numerical simulation of the stochastic Gross-Pitaevskii equation.

We also undertook a numerical investigation of damping and the role of stochastic noise in dark soliton decay, producing a number of interesting results. Noise induced decoupling of solitons was witnessed in a periodic system. Noise induced decay of a stationary soliton in a finite homogenous system was also observed. Most interestingly, we found that the thermal noise caused a temperature dependent reduction of the damping. This result contrasts vortex behaviour found by Rooney *et al.* [38]. The averaged stochastic results approached the dissipative Gross-Pitaevskii equation in the limit of zero temperature. The results show there is still some interesting phenomena yet to be fully understood, even for a single soliton on an infinite background, when stochastic noise is considered. There is clear motivation for providing an approximate variational method that includes noise for a dark soliton system.

8.1 Directions for Future Research

The work has highlighted the importance of noise on the decay of dark solitons in a warm Bose-Einstein condensate. The success of describing damping via the perturbative Lagrangian variational technique, as well as the potential of more complete variational techniques, indicate that results should be possible. The next step is to consider explicitly finite systems, where the effects of the noise on the background will be bounded.

In terms of numerical investigation, future research could attempt to investigate the noise dependence of damping in more detail, to shed more light on the effects of sound emission and fluctuations from the pure soliton form.

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