Kohn Mode Damping in a Bose-Einstein Condensate

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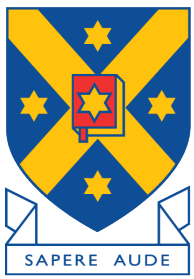

Abstract

The Kohn modes of the groundstate of the Bose-Einstein condensate were observed and equations describe its motion in the high temperature and low termperature regimes were considered. At least a partially complete theory and equation of motion was derived but requires more testing and consideration of wider regimes of both temperature changes and spatial changes in the degree of confinement of the condensate. The analytical expression of the damping rate was found to closely match that obtained numerically through the 1D SPGPE, and formal solution of the damped finite temperature Kohn mode was obtained.

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Chapter 1

Introduction

Bose-Einstein condensates (BEC) have a vast number of both interesting and unusual properties. Since the first creation of one in 1995, many of these phenomena are by now quite well understood, for instance soliton matter wave disturbances, quantized vortices, collective excitations and many more. Furthermore there has been many studies into the Kohn modes of trapped BECs and the investigations of the Kohn theorem in each of the physical situations and studies. It is also well-known that at low temperatures, the Gross-Pitaevskii Equation (GPE) is a good description of the condensate dynamics, but at higher temperatures it is insufficient and the coupling to the thermal cloud makes the situation more complicated, thus requiring the inclusion of noise terms and spontaneous processes. Much active research has gone into developing the theory describing high temperature BECs, for instance C-field theory. [5]

1.1 The Physical Problem

This research thesis looks into mainly the energy damping effects on a particular BEC mode due to the interaction between a BEC and a thermal cloud in situation of high temperature. In particular the ground state of a cigar-shaped BEC superfluid is considered, which is immersed in a thermal cloud of distinct component, i.e. both the condensate and the thermal cloud are made up of distinct species with roughly equal masses, because the rates are known analytically. The situation is that the transverse confinement is quite loose (relative to component $|2\rangle$), giving one-dimensional condensate motion in the midst of a three-dimensional thermal cloud. The regime of the 3D thermal cloud makes the problem tractable as the consequent reservoir interactions are simpler, as is the consideration of an effective 1D condensate motion [22]. In this setup the whole system is trapped by a harmonic potential and the interactions between the condensate and the thermal cloud is such that the there is no change in

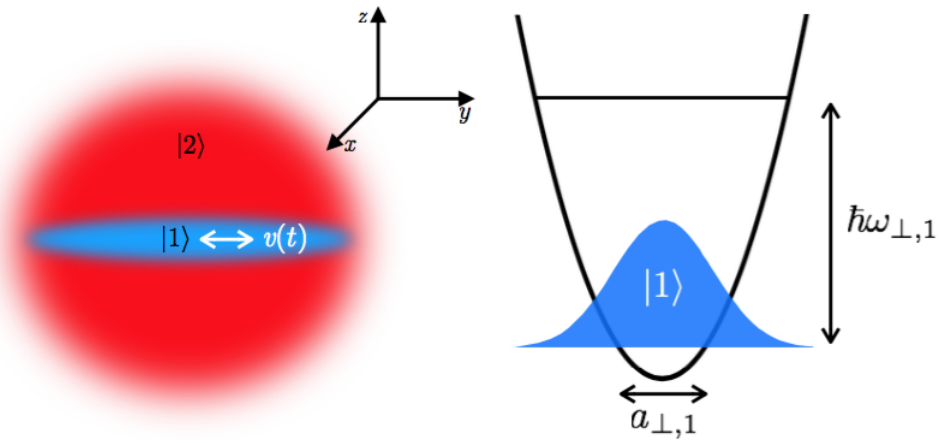


Figure 1.1: The general setup to illustrate the high temperature case we are considering. Here we have the 3D thermal cloud $|2\rangle$ surrounding the 1D BEC $|1\rangle$ which can move in the thermal cloud thus interacting with it. The trapping potential is just the 1D harmonic oscillator one.

the number of particles in each component, i.e. there is no number damping, only transfer of energy or energy damping happens. The exclusion of number damping is reasonable due to reasons pertaining to spin-changing collisions, and for $T \ll T_c$, energy damping is dominant. [6]. Furthermore the temperature regime considered here is one that is much lower than the critical temperature T_c of component 1, denoted $|1\rangle$ but higher than the critical temperature of component 2, denoted $|2\rangle$, so that this component is non-condensed.

The dynamics of the condensate density at this high temperature regime, i.e., $T_{c,2} < T \ll T_{c,1}$, is described by a modified GPE which is an extension of the Projected Gross-Pitaevskii Equation (PGPE) and called the Stochastic Projected Gross-Pitaevskii Equation (SPGPE) — a technique obtained from C-field theory. The SPGPE is essentially a stochastic differential equation suited for high temperature problems involving reservoir interactions and number transfer, i.e. open quantum systems. In the present context, we start with the general 3D SPGPE, put on appropriate confinement geometry, see Figure (1.1) and thereby reduce it to an effectively 1D equation describing the consequent C-field behaviour. It is helpful to note that the SPGPE has similar dynamical behaviour as the bare GPE (of course plus additional features).

A Bose-Einstein condensate trapped in a harmonic potential is known to oscillate along many different modes. [8] These include for example rigid sloshing of the condensate column density about the center of the trap (Kohn mode), column density oscillations about the chemical potential (breathing mode) —also known as collective modes, and oscillations of the condensate with respect to the transverse plane when the trapping potential is slightly rotated suddenly (scissors mode) [17]. The ground state of the BEC is studied, which is ob-

tained via the Thomas-Fermi approximation and the Kohn mode effects are studied in the presence of energy damping and quantum fluctuations and noise using the SPGPE. This will allow one to obtain the equations of motion of the condensate in this high temperature open quantum system. In the past much research has gone into number damping or BEC growth processes, and not much into the energy damping reservoir interactions. [7] The overall aim of this research project is to (i) develop an analytical model to describe the motion of the Kohn mode and other information relating to it in the described physical context, and (ii) compare this with the numerical modelling and calculations and interpret and learn more about energy damping and observe the reservoir interactions.

1.2 Bose-Einstein Condensation

When an ultra-dilute gas of weakly-interacting bosons, which are indistinguishable particles of integer spin and symmetric quantum states(or wavefunctions), are cooled to sufficiently low temperatures past a characteristic (well-defined) temperature then all the particles collapse into a single quantum state. This is possible for bosons, which are unlike fermions (half-integer spin and antisymmetric quantum states) that are restrained by the Pauli Exclusion Principle. Thus the true ground state is macroscopically occupied, the system has reached Bose-degeneracy and a new state of matter emerges, the onset of which is characterised by a discontinuous phase transition. This interesting state of matter is the Bose-Einstein condensate and was first predicted by Einstein in 1924 after applying Bose-Einstein statistics in analysing a gas of bosons. It is important to note that the thermodynamic phase transition is driven by quantum particle statistics, i.e. Bose-Einstein statistics, and not by the interactions of the particles. [3]

At high temperatures, a dilute gas of particles behave classically since their de Broglie wavelength is much smaller than the average distance between the particles and thus their trajectories can be described deterministically. However as the temperature is lowered, the de Broglie wavelength of the atoms becomes larger and past the critical temperature they become comparable to the inter-particle spacing and they overlap to achieve quantum degeneracy where all the particles are in the ground state. This means that we can describe the system with a single macroscopic wavefunction. The temperatures required to achieve Bose-Einstein condensation range 0.5 μK to 2 μK and ultra-low density vapours about 1/100,000 the density of air. The latter is so to limit three-body collision rate ($R_{\text{3bc}} \propto n^2$, where n is the density) compared with binary or two-body collision rate (which are ideal because they are elastic in nature and $R_{\text{2bc}} \propto n$), because three-body collisions produce particle clustering that can impede BEC formation. Once this is achieved, then the use of various techniques such as laser cooling and evaporative cooling, in trapping potentials like magneto-optical traps (MOT)

allows for the low temperatures needed for BEC formation. [2]

The first BEC was created in 1995, with a dilute vapour of Rubidium-87 atoms. The signature onset of condensation is the sharp peak of the momentum space density seen in the familiar absorption image as the temperature is lowered from above the critical temperature to below the critical temperature. Here the number of particles with lowest speed increases as the temperatures dips lower and lower than the critical temperature by means of succession of evaporative cooling, letting the higher energy atoms to escape the trap. BECs have been the subject of much intense research both experimental and theoretical and are highly important laboratories for ultra-low temperature condensed matter physics experiments. [3] [2]

1.3 The Kohn Theorem

The original Kohn theorem was first discovered by W. Kohn in 1961 in the context of three-dimensional bulk electron gas with short-range interactions. [21] He noted that the cyclotron resonance frequency of the electron gas was independent of the electron-electron interactions. Later this was extended further to atoms of a harmonically trapped gas by Dobson [10].

1.4 The Kohn Mode

The Kohn mode is an interesting mode of oscillation of the BEC density characterised by rigid sloshing about the centre of the harmonic trap. It is induced by a kick giving the condensate an initial velocity. The Kohn mode is the oscillation of the centre of the mass position of the condensate at the characteristic trap frequency and is independent of the internal inter-particle interactions and hence separates out as a solution to the GPE and is present at even high temperatures as it must as a consequence of the Kohn theorem. [24][19][1] Consider the Hamiltonian:

$$\hat{\mathcal{H}} = \sum_{i=1}^{N} \left[\frac{p_i^2}{2m} + \frac{1}{2} m(\omega_x^2 x_i^2 + \omega_y^2 y_i^2 + \omega_z^2 z_i^2) \right] + \sum_{i<j} U(|\mathbf{x}_i - \mathbf{x}_j|), \tag{1.1}$$

where we consider a general anisotropic harmonic trap, and U is the inter-particle interaction potential and we take a pair-wise sum for an N-particle bosonic system. Next we define the centre of mass position operator as:

$$\hat{X}_a \equiv \frac{1}{N} \sum_{i=1}^{N} r_{ia}, \tag{1.2}$$

where the positions of the particles are given by r_{ia}. Next writing the Heisenberg equation of motion for the centre of mass position operator we get:

$$\frac{d\hat{X}_a}{dt} = -\frac{i}{\hbar}[\hat{X}_a, \hat{H}] = \frac{\hat{P}_a}{M}, \tag{1.3}$$

where

$$\hat{P}_a = \sum_{i=1}^{N} p_{ia}, \text{ and } M = Nm. \tag{1.4}$$

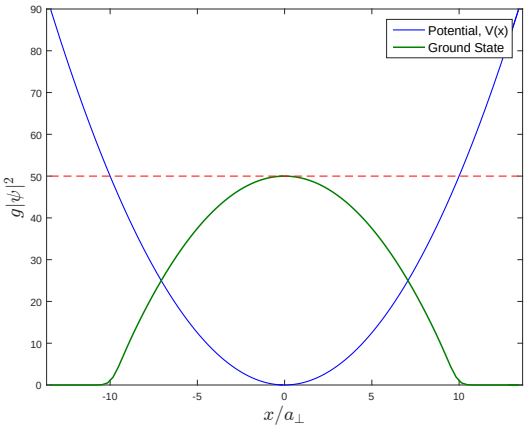
Here $\hat{P}_a$ is the total momentum operator and M is the total mass of the system. From the Heisenberg equation above we get the following well-known differential equation for the centre of mass operator:

$$\frac{d^2\hat{X}_a}{dt^2} + \omega_a^2 \hat{X}_a = 0. \tag{1.5}$$

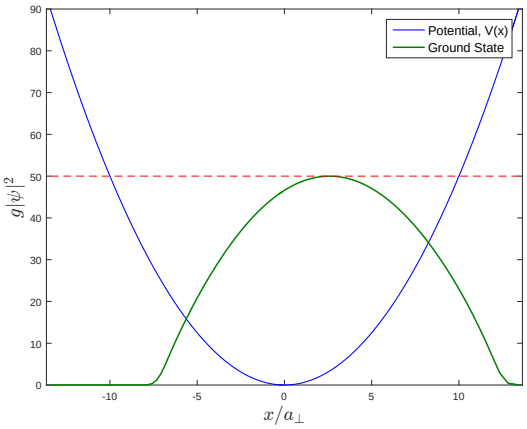
With sinusoidal general solution given by:

$$\hat{X}_a = \hat{X}_a(0)\cos\omega_a t + \frac{1}{\omega_a}\frac{d\hat{X}_a(0)}{dt}\sin\omega_a t. \tag{1.6}$$

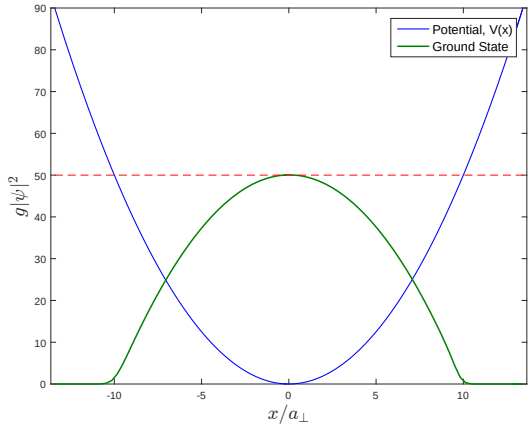
That is the centre of mass oscillates with amplitude $\hat{A}_0$ and $\hat{B}_0$ and characteristic frequency of the trap ω_a. This is the BEC Kohn mode with no damping processes as was described above. The Kohn mode of oscillation is shown in Fig. 1.2 over one trapcycle.



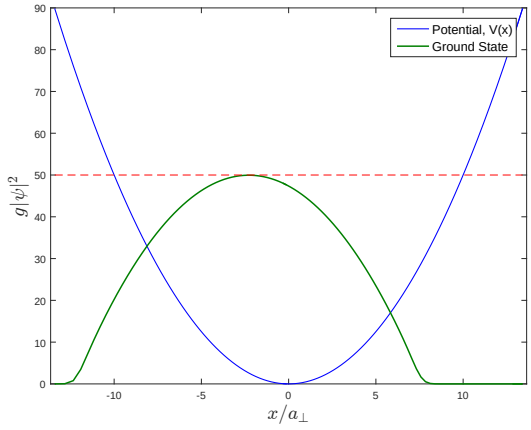
(a) $t = 0\ \omega_x^{-1}$



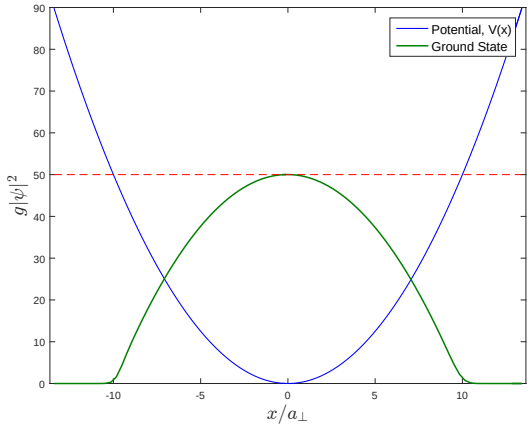
(b) $t = 0.25\ \omega_x^{-1}$



(c) $t = 0.5\ \omega_x^{-1}$



(d) $t = 0.75\ \omega_x^{-1}$



(e) $t = 1.0\ \omega_x^{-1}$

Figure 1.2: The Kohn mode shown over one trapcycle.

Chapter 2

Theoretical Description of the Bose-Einstein Condensate

In this chapter we look at the mathematical description of the BEC, that is the physical theory describing the behaviour and dynamics of BECs. It will be shown that there are many theories pertaining to different physical regimes, the most well-known is that for a pure condensate dynamics: the Gross-Pitaevskii Equation, which is obtained from the application of mean-field theory. Beyond this there is the theory of condensates at finite (higher) temperatures, and which deals with the coupling between the condensate fraction and the thermal cloud fraction and involving quantum and thermal processes. We describe effective field theory and thus C-field theory and show how this yields new GPE-like equations relevant to the context of this project.

2.1 The Gross-Pitaevskii Equation

The Gross-Pitaevskii Equation is a simple and effective non-linear Schrödinger equation describing the time evolution of a low temperature system of interacting bosons. Then consider this system for which the many-body Hamiltonian is defined by the second-quantised form:

$$\hat{\mathcal{H}} = \int d\mathbf{x}\, \hat{\Psi}^\dagger(\mathbf{x}) \hat{\mathcal{H}}_{sp} \hat{\Psi}(\mathbf{x}) + \frac{1}{2} \iint d\mathbf{x} d\mathbf{x}' \hat{\Psi}^\dagger(\mathbf{x}) \hat{\Psi}^\dagger(\mathbf{x}') U(\mathbf{x}-\mathbf{x}') \hat{\Psi}(\mathbf{x}') \hat{\Psi}(\mathbf{x}). \tag{2.1}$$

Where $U(\mathbf{x}-\mathbf{x}')$ is the inter-particle interaction, with $\hat{\Psi}(\mathbf{x})$ and $\hat{\Psi}^\dagger(\mathbf{x})$ being the Bose field operators which act to destroy and create a boson at $\mathbf{x}$. $\hat{\mathcal{H}}_{\rm sp}$ is the single particle Hamiltonian given by:

$$\hat{\mathcal{H}}_{\rm sp} = -\frac{\hbar^2}{2m}\nabla^2 + V_{\rm ext}(\mathbf{x}) \tag{2.2}$$

where also $V_{\rm ext}(\mathbf{x})$ is the external trapping potential confining this system. We now apply principles of mean-field theory to side-step the issue of solving the complete many-body Schrödinger equation exactly. But it turns out that the mean-field method is quite useful in predicting the static, dynamic and thermodynamic properties of trapped Bose gases [8]. Mean-field theory basically says that we can separate out the condensate contribution to the field operator. For instance the field operator can be written as:

$$\hat{\Psi}(\mathbf{x}) = \sum_{\alpha} \hat{a}_{\alpha} \Psi_{\alpha}(\mathbf{x}), \tag{2.3}$$

where $\Psi_{\alpha}(\mathbf{x})$ is the single-particle wavefunctions and $\hat{a}_{\alpha}$ are the corresponding annihilation operators. We can define fock states which are number states that are the eigenstate of the number operator $\hat{n}_{\alpha} = \hat{a}^{\dagger}_{\alpha}\hat{a}_{\alpha}$, and the actions of the creation and annihilation operator on these states are:

$$\hat{a}^{\dagger}_{\alpha} \left|n_0, n_1, ..., n_{\alpha}, ...\right\rangle = \sqrt{n_{\alpha}+1} \left|n_0, n_1, ..., n_{\alpha}+1, ...\right\rangle, \tag{2.4}$$

$$\hat{a}_{\alpha} \left|n_0, n_1, ..., n_{\alpha}, ...\right\rangle = \sqrt{n_{\alpha}} \left|n_0, n_1, ..., n_{\alpha}-1, ...\right\rangle. \tag{2.5}$$

With these operators satisfying the boson commutation relations:

$$[\hat{a}_i, \hat{a}^{\dagger}_j] = \delta_{i,j}\ , \quad \text{and} \quad [\hat{a}_i, \hat{a}_j] = [\hat{a}^{\dagger}_i, \hat{a}^{\dagger}_j] = 0. \tag{2.6}$$

Now then we consider the case of significant Bose-degeneracy where the number of particles in a given state, say the ground state is very large so that this state is macroscopically occupied, then $n_0 \gg 1$ so that $\lim_{N\to\infty}(n_0/N)$ is finite. Subsequently it can be seen quite clearly that as $N \to \infty$ the effect of the creation and annihilation operators are negligible as the result in Eq. (2.4) and Eq. (2.5) remain the same setup, and so we can make the replacement: $\hat{a}_0 \to c$ and $\hat{a}^{\dagger}_0 \to c$, where $c = \sqrt{n_0}$ is the same *number*. Then the field operator can be written in terms of a mean field plus a fluctuating perturbation, $\tilde{\Psi}(\mathbf{x}, t)$:

$$\hat{\Psi}(\mathbf{x}) = \sqrt{\frac{n_0}{V}} + \hat{\Psi}'(\mathbf{x}) \tag{2.7}$$

But in general we can write the field operator for the time-dependent case as:

$$\hat{\Psi}(\mathbf{x}, t) = \Psi(\mathbf{x}, t) + \tilde{\Psi}(\mathbf{x}, t)\ , \quad \text{such that} \quad \langle\hat{\Psi}(\mathbf{x}, t)\rangle = \Psi(\mathbf{x}, t). \tag{2.8}$$

Here $\Psi(\mathbf{x}, t)$ is the BEC wavefunction or the order parameter, and because of the second statement of Eq. (2.8) it is a classical field. Now in order to proceed further we write down

the Heisenberg equation of motion for the field operator using the Hamiltonian, Eq. (5.1):

$$i\hbar\frac{\partial\hat{\Psi}(\mathbf{x},t)}{\partial t} = [\hat{\Psi}(\mathbf{x},t),\hat{H}] \tag{2.9}$$

$$= \left(-\frac{\hbar^2\nabla^2}{2m} + V_{\text{ext}}(\mathbf{x}) + \int d\mathbf{x}'\ \hat{\Psi}^\dagger(\mathbf{x}',t)U(\mathbf{x}'-\mathbf{x})\hat{\Psi}(\mathbf{x}',t)\right)\hat{\Psi}(\mathbf{x},t) \tag{2.10}$$

For a dilute, low-temperature weakly interacting gas with a significant degree of Bose-degeneracy or large (ground state) occupation numbers we can make the mean-field replacement: $\hat{\Psi}(\mathbf{x},t) \to \Psi(\mathbf{x},t)$, since we are dealing with a classical field with negligible quantum fluctuations and we can assume that all the particles are in a single mode. Furthermore in this physical regime only two-body collisions are significant (since ultra-low density and low energy) and these only depend on the characteristic length, say a_s, and we can make a further replacement for the two-body interaction potential:

$$U(\mathbf{x}'-\mathbf{x}) \to g\delta(\mathbf{x}'-\mathbf{x})\ , \quad \text{where} \quad g = \frac{4\pi\hbar^2 a_s}{m}. \tag{2.11}$$

The constant g is the particle interaction strength or the coupling constant and a_s is the s-wave scattering length describing low-energy scattering. With all these replacements we finally get:

$$i\hbar\frac{\partial\Psi(\mathbf{x},t)}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{\text{ext}}(\mathbf{x}) + g|\Psi(\mathbf{x},t)|^2\right)\Psi(\mathbf{x},t). \tag{2.12}$$

Eq. (2.12) is the Gross-Pitaevskii Equation or GPE and it describes the dynamics of pure BECs at zero temperature very well. We see that this is just the time-dependent Schrödinger equation plus a non-linear term due to the interactions between the particles. Depending on the types of inter-particle interactions we can, through a_s, have either $g < 0$ or we can have $g > 0$ corresponding to attractive and repulsive interactions respectively. [4]

2.2 Thomas-Fermi Approximation

When the total number of particles N is very large, the interactions between the particles are prominent and the kinetic energy of the particles is negligible and hence the kinetic energy term in the GPE can be dropped. This is the essence of the Thomas-Fermi approximation. If we further take as our product ansatz: $\Psi(\mathbf{x},t) = \exp(-i\mu_n t/\hbar)\psi_n(\mathbf{x})$ and put this into the GPE above we get:

$$\mu\exp\left(-\frac{i\mu}{\hbar}t\right)\psi(\mathbf{x}) = \left(V_{\text{ext}}(\mathbf{x}) + g|\psi(\mathbf{x},t)|^2\right)\exp\left(-\frac{i\mu}{\hbar}t\right)\psi(\mathbf{x}), \tag{2.13}$$

Which is really the time-independent GPE, then we take the limit as $Ng/4\pi a_{\text{ho}}^3\hbar\omega_{\text{ho}} \to \infty$, (for $\omega_{\text{ho}} = (\bar{\omega}_x\bar{\omega}_y\bar{\omega}_z)^{1/3}$ and thus $a_{\text{ho}} = \sqrt{\hbar/m\omega_{\text{ho}}}$), to get: [20] [23]

$$V_{\text{ext}}(\mathbf{x}) + g|\psi(\mathbf{x},t)|^2 = \mu, \tag{2.14}$$

Hence we get the Thomas-Fermi (TF) ground state wavefunction of the BEC:

$$\psi_{TF}(\mathbf{x},t) = \sqrt{\frac{\mu - V_{\text{ext}}(\mathbf{x})}{g}}\,. \tag{2.15}$$

Then for a harmonic 1D potential, $V_{\text{ext}}(x) = \frac{1}{2}m\omega_x^2x^2$, this result can be further reduced to:

$$\psi_{TF}(x,t) = \sqrt{\frac{\mu}{g}\left(1 - \frac{x^2}{R^2}\right)}\,, \quad \text{where} \quad R \equiv \sqrt{\frac{2\mu}{m\omega_x^2}}\,. \tag{2.16}$$

Here R is the 'TF radius' or the roots of the TF wavefunction. The TF condensate density is an inverted parabola:

$$n_{TF} = |\psi_{TF}(x,t)|^2 = \frac{\mu}{g}\left(1 - \frac{x^2}{R^2}\right) = n_0\left(1 - \frac{x^2}{R^2}\right). \tag{2.17}$$

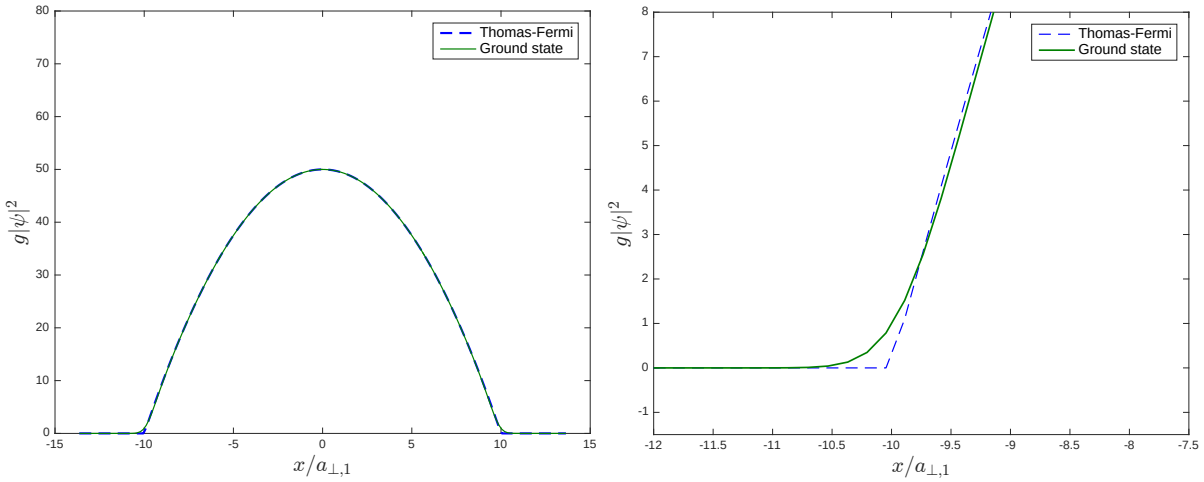
In fact the it is a very good approximation to the true ground state density (and thus wavefunction) except at the edges where it beaks down (it is not differential there, i.e. at $x = \pm R$). With peak density $n_0 = \mu/g$ at the centre of the trap. This is plotted in Fig. (2.1(a)-2.1(b)).

2.3 Effective One-Dimensional Systems

The three-dimensional GPE is much more complicated than in lower dimensions. So it would be ideal if we can apply some sort of confinement geometry to reduce a higher-dimensional problem into a lower-dimensional one. Consider the regime where the external trapping potential has strong parabolic confinement along the transverse axes:

$$V_1(\mathbf{x}) = V_0(x) + \frac{1}{2}m\omega_\perp^2(y^2+z^2) = \frac{1}{2}m\omega_x^2x^2 + \frac{1}{2}m\omega_\perp^2r_\perp^2. \tag{2.18}$$

For transverse plane given by: $r_\perp = \sqrt{y^2+z^2}$. It is also important to treat the thermal cloud or reservoir as being 3D otherwise we get a much more difficult analysis due to the change in the dispersion. This allows for an effective 1D dynamics of the first component and the simpler well-known form of the 3D reservoir interaction. [22] A valid quasi-1D description



(a) (b)

Figure 2.1: Plots showing the Thomas-Fermi approximation to the groundstate: (a) here the TF groundstate (dashed) is plotted with the true groundstate; (b) the close up of the edge, showing the divergence of the approximation.

for the superfluid motion with a 3D reservoir cloud theory is established by the condition that:

$$\mu \ll \hbar\omega_\perp < k_{\mathrm{B}}T, \tag{2.19}$$

where μ is the 1D chemical potential. For this 1D approximation we have tight transverse confinement: $a_\perp \ll a_x$ or equivalently $\omega_x \ll \omega_\perp$, for the characteristic lengths defined as:

$$a_\perp \equiv \sqrt{\frac{\hbar}{m\omega_\perp}}, \quad \text{and} \quad a_x \equiv \sqrt{\frac{\hbar}{m\omega_x}}. \tag{2.20}$$

The final result is then that we have the BEC moving only along one particular axis, here the x-axis. This dimensional reduction further changes the interaction parameter, giving $g = g_{1\mathrm{D}} = 2\hbar\omega_\perp a_s$. These treatments give the set up shown in the previous chapter and also in Figure (2.2) where we have the resulting cigar-shaped prolate geometry in which the superfluid moves along one-dimension immersed in the broad thermal cloud. [7]

2.4 High-Temperature BEC theory

To understand the behaviour of BECs at higher temperatures approaching T_c, that is $T \sim T_c$, we must step away from the bare GPE and instead develop a theory that takes into account the fact that at these real-life experimental temperatures, there is not only a condensate fraction

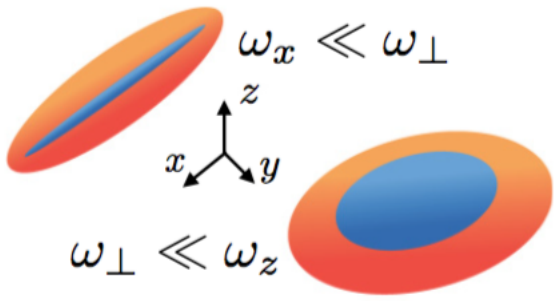


Figure 2.2: The low-dimensional reduction scheme in a 3D thermal cloud. To the top left we have the situation where tight transverse confinement means effective 1D condensate motion along one axis—this is the scheme considered in this thesis. However to the bottom right, the scheme is loose transverse confinement and an oblate geometry and effective 2D description. Taken from [7]

N_0 present, but also a incoherent thermal fraction $N - N_0$ for a total particle number, say N. At finite temperatures there is coupling or interaction between the condensate fraction, component 1 $|1\rangle$ and the broad thermal cloud of component 2, $|2\rangle$ and the result is scattering and spontaneous quantum phenomena.

It turns out that warm BECs can be well-described by a stochastic differential equation that has similar dynamics as the GPE but contains new terms pertaining to processes mentioned above, namely growth and scattering processes. This is the stochstic projected GPE which is a treatment within the C-field formalism. [5]

2.5 C-field Theory

It would be useful to somehow mediate both coherent and incoherent physics in a single theory so to systematically explain the processes that occur at finite temperatures. This can be done with a theoretical technique called C-field theory. It relies on the quantum system having high occupation numbers and modes that are significantly Bose-degenerate, thereby describing this system with a classical field, and hence why the theory is called *C*-field. Naturally this theory considers the separation of the system into two principle regions: the *coherent* and *incoherent* regions. This is done by means of a well-defined energy cutoff. By the end of this section, we will have obtained a number of C-field techniques and results which are relevant to different physical regimes, in particular the SDE describing the full behaviour of a warm BEC: the Stochastic Projected GPE or SPGPE. [5] [9]

2.5.1 Effective Field Theory

As a start we first consider a low energy subspace **L** and write the Hamiltonian for this system of dilute gas of bosons with contact interactions. Thus consider this Hamiltonian in terms of field operators:

$$\hat{\mathcal{H}} = \int d^3\mathbf{x}\, \hat{\Psi}^\dagger(\mathbf{x})\hat{\mathcal{H}}_{\rm sp}\hat{\Psi}(\mathbf{x}) + \frac{1}{2}\iint d^3\mathbf{x}d^3\mathbf{x}'\hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}^\dagger(\mathbf{x}')U(\mathbf{x}-\mathbf{x}')\hat{\Psi}(\mathbf{x}')\hat{\Psi}(\mathbf{x}). \tag{2.21}$$

Where the single particle Hamiltonian is:

$$\hat{\mathcal{H}}_{\rm sp} = \mathcal{H}_0 + \delta V(\mathbf{x},t) = -\frac{\hbar^2\nabla^2}{2m} + V_0(\mathbf{x}) + \delta V(\mathbf{x},t) \tag{2.22}$$

And the last term is a perturbation Hamiltonian, and where the single-particle eigenstates $\phi_n(\mathbf{x})$ solve the eigenvalue equation:

$$\hat{\mathcal{H}}_0\phi_n(\mathbf{x}) = \epsilon_n\phi_n(\mathbf{x}) \tag{2.23}$$

Effective field theory then says that since the characteristic length of the interaction $U(\mathbf{x})$, say r_0,which depends on the particle separation, is much smaller than the characteristic lengths encountered in many condensed matter physics experiments, it would be better to replace this r_0 with a larger, more representative characteristic length. It turns out that it would be more appropriate to take this to be the s-wave scattering length.

We proceed then and reconsider the low-energy subspace, defined as being spanned by single-particle states of energy less than a certain $E_{\rm max}$, that is: $\mathbf{L} = \{n : \epsilon_n \leq E_{\rm max}\}$. The $E_{\rm max}$ value must be within a particular band of values, for instance we remove the high energy components that correspond to the range of the interaction potential and retain instead those lower energy ones that correspond to the larger characteristic length Γ (which gives an upper bound for $E_{\rm max}$). Furthermore we require $E_{\rm max} \gg k_{\rm B}T$ and $E_{\rm max} \gg \mu$ in order that the states removed cannot be occupied by thermal and interaction influences. Specifically we have that all states with momentum greater than the value:

$$\hbar\Lambda(\mathbf{x}) \cong \sqrt{2m(E_{\rm max} - V_0(\mathbf{x}))}, \tag{2.24}$$

are eliminated, resulting in what is referred to as coarse-graining to a larger characteristic length of $\Gamma = 1/\Lambda(\mathbf{x})$, where $\Lambda(\mathbf{x})$ is the maximum wave number.

Then the resulting effective Hamiltonian that we get is, expressed in terms of the coarse-

grained field operators, $\hat{\psi}(\mathbf{x})$ and $\hat{\psi}^{\dagger}(\mathbf{x})$:

$$\hat{\mathcal{H}}_{\text{eff}} = \int d^3\mathbf{x}\hat{\psi}^{\dagger}(\mathbf{x})\hat{\mathcal{H}}_{\text{sp}}\hat{\psi}(\mathbf{x}) + \frac{u}{2}\int d^3\mathbf{x}\hat{\psi}^{\dagger}(\mathbf{x})\hat{\psi}^{\dagger}(\mathbf{x})\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{x}). \tag{2.25}$$

The coarse-grained field operators only contain the modes in $\mathbf{L}$:

$$\hat{\psi}(\mathbf{x}) = \sum_{n\in\mathbf{L}} \hat{a}_n\phi_n(\mathbf{x}) \quad \text{and} \quad \hat{\psi}^{\dagger}(\mathbf{x}) = \sum_{n\in\mathbf{L}} \hat{a}_n^{\dagger}\phi_n^*(\mathbf{x}), \tag{2.26}$$

and their commutator is equal to the coarse-grained delta function:

$$[\hat{\psi}(\mathbf{x}), \hat{\psi}^{\dagger}(\mathbf{x}')] = \delta_{\mathbf{L}}(\mathbf{x}-\mathbf{x}'). \tag{2.27}$$

And also the creation and annihilation operators $\hat{a}^{\dagger}$, $\hat{a}$ satisfy the usual Bose commutation relations as in Eq.(2.6). Using all the above, we can then arrive at the following Heisenberg equation of motion:

$$i\hbar\frac{\partial\hat{\psi}(\mathbf{x})}{\partial t} = \int d^3\mathbf{x}'\,\delta_{\mathbf{L}}(\mathbf{x}-\mathbf{x}')\left(\hat{\mathcal{H}}_{\text{sp}}\hat{\psi}(\mathbf{x}') + u\hat{\psi}^{\dagger}(\mathbf{x}')\hat{\psi}(\mathbf{x}')\hat{\psi}(\mathbf{x})\right). \tag{2.28}$$

Which is an equation of motion for the coarse-grained field operator over the well-defined subspace $\mathbf{L}$. This can be further formulated for a number of physical regimes of interest as will be now shown in the next section. We then use C-field techniques to obtain a more numerically-illuminating result.

2.5.2 Projection onto the C-field

We have already seen in the previous section and in particular Eq.(2.26), that we only consider all the modes in $\mathbf{L}$. But this is still quite large to manage and we should be able to split this up into smaller 'sub-subspaces'. Thus we define two important regions of interest: (i) The *C-field region* or the *coherent region* ($\mathbf{C}$), which holds all the lowest energy highly-occupied modes and describes the condensate (hence coherent), and (ii) the *Incoherent region* ($\mathbf{I}$) which holds all the other higher-energy modes in $\mathbf{L}$.

So basically we have $\mathbf{L} = \mathbf{C} + \mathbf{I}$ and the definitions: $\mathbf{C} = \{n : \epsilon_n \leq E_{\text{cut}}\}$ and $\mathbf{I} = \{n : E_{\text{cut}} < \epsilon_n \leq E_{\text{max}}\}$. These definitions involve the single-particle energy E_{cut}, where single-particle modes having the energies $\epsilon \leq E_{\text{cut}}$ span the sub-subspace* $\mathbf{C}$ and those with energies $E_{\text{cut}} < \epsilon \leq E_{\text{max}}$ span $\mathbf{I}$. Furthermore the orthogonal projection operators are

defined in general for a function $f(\mathbf{x})$ as:

$$\hat{\mathcal{P}}_{\mathbf{C}}\{f(\mathbf{x})\} \equiv \sum_{n\in\mathbf{C}} \phi_n(\mathbf{x}) \int d^3\mathbf{x}' \phi_n^*(\mathbf{x}') f(\mathbf{x}'), \tag{2.29}$$

and

$$\hat{\mathcal{P}}_{\mathbf{I}}\{f(\mathbf{x})\} \equiv \sum_{n\in\mathbf{I}} \phi_n(\mathbf{x}) \int d^3\mathbf{x}' \phi_n^*(\mathbf{x}') f(\mathbf{x}'). \tag{2.30}$$

We note the result that:

$$\hat{\mathcal{P}}_{\mathbf{I}}\hat{\mathcal{P}}_{\mathbf{C}} = 0, \;\; \hat{\mathcal{P}}_{\mathbf{C}}\hat{\mathcal{P}}_{\mathbf{C}} = \hat{\mathcal{P}}_{\mathbf{C}}, \;\; \hat{\mathcal{P}}_{\mathbf{I}}\hat{\mathcal{P}}_{\mathbf{I}} = \hat{\mathcal{P}}_{\mathbf{I}}, \;\; \text{and} \;\; [\hat{\mathcal{P}}_{\mathbf{C}}, \hat{\mathcal{P}}_{\mathbf{I}}] = 0. \tag{2.31}$$

Furthermore, we now split the total field operator into the coherent and incoherent field operators via projections onto the respective regions:

$$\hat{\psi}(\mathbf{x}) = \hat{\psi}_{\mathbf{C}}(\mathbf{x}) + \hat{\psi}_{\mathbf{I}}(\mathbf{x}), \tag{2.32}$$

where we have

$$\hat{\psi}_{\mathbf{C}}(\mathbf{x}) \equiv \hat{\mathcal{P}}_{\mathbf{C}}\{\hat{\psi}(\mathbf{x})\} = \sum_{n\in\mathbf{C}} \hat{a}_n \phi_n(\mathbf{x}), \tag{2.33}$$

$$\hat{\psi}_{\mathbf{I}}(\mathbf{x}) \equiv \hat{\mathcal{P}}_{\mathbf{I}}\{\hat{\psi}(\mathbf{x})\} = \sum_{n\in\mathbf{I}} \hat{a}_n \phi_n(\mathbf{x}). \tag{2.34}$$

Which will give more meaning to the theory since now we can treat each region separately in terms of these 'sub-field operators'. It's also a good idea to consider the commutator of the C-field field operators:

$$[\hat{\psi}_{\mathbf{C}}(\mathbf{x}), \hat{\psi}_{\mathbf{C}}^\dagger(\mathbf{x}')] = \delta_{\mathbf{C}}(\mathbf{x} - \mathbf{x}'), \tag{2.35}$$

where

$$\delta_{\mathbf{C}}(\mathbf{x} - \mathbf{x}') \equiv \sum_{n\in\mathbf{C}} \phi_n(\mathbf{x}) \phi_n^*(\mathbf{x}'), \tag{2.36}$$

this $\delta_{\mathbf{C}}(\mathbf{x} - \mathbf{x}')$ takes the role of the usual Dirac delta-function for the **C** region describing the condensate.

2.5.3 The Hamiltonian and Equation of Motion

Like we did above with the field operators, the effective Hamiltonian of Eq.(2.25) can be expressed in terms of $\hat{\psi}_{\mathbf{C}}(\mathbf{x})$ and $\hat{\psi}_{\mathbf{I}}(\mathbf{x})$:

$$\hat{\mathcal{H}}_{\text{eff}} = \hat{\mathcal{H}}_{\mathbf{C}} = \hat{\mathcal{H}}_{\mathbf{C}} + \hat{\mathcal{H}}_{\mathbf{I}} + \hat{\mathcal{H}}_{\mathbf{I},\mathbf{C}} \;, \tag{2.37}$$

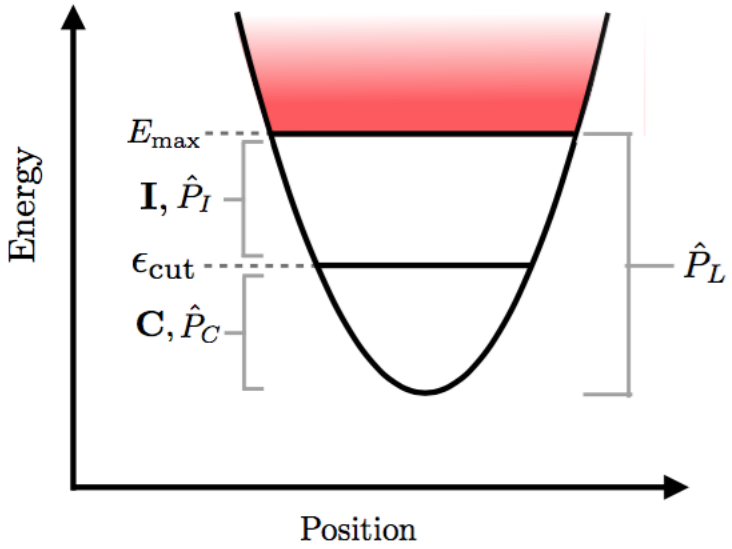


Figure 2.3: Simple schematic showing the c-field region, $\mathbf{C}$, incoherent region $\mathbf{I}$ and also the high energy states that are eliminated. We note that the c-field particles are described by a quantum description using classical fields and the incoherent region particles are treated semi-classically.

here $\hat{\mathcal{H}}_{\mathbf{C}}$ denotes the Hamiltonian term containing only coherent field operators, and likewise $\hat{\mathcal{H}}_{\mathbf{I}}$ denotes only incoherent field operators and for the last term we have:

$$\hat{\mathcal{H}}_{\mathbf{I},\mathbf{C}} = \hat{\mathcal{H}}^{(1)}_{\mathbf{I},\mathbf{C}} + \hat{\mathcal{H}}^{(2)}_{\mathbf{I},\mathbf{C}} + \hat{\mathcal{H}}^{(3)}_{\mathbf{I},\mathbf{C}} \,. \tag{2.38}$$

This involves three cross-terms of the various field operators representing the interaction between the atoms in the different regions. Writing them out we then have: [13]

$$\hat{\mathcal{H}}_{\mathbf{C}} = \int d^3\mathbf{x}\hat{\psi}^\dagger_{\mathbf{C}}(\mathbf{x})\hat{\mathcal{H}}_{\text{sp}}\hat{\psi}_{\mathbf{C}}(\mathbf{x}) + \frac{g}{2}\int d^3\mathbf{x}\hat{\psi}^\dagger_{\mathbf{C}}(\mathbf{x})\hat{\psi}^\dagger_{\mathbf{C}}(\mathbf{x})\hat{\psi}_{\mathbf{C}}(\mathbf{x})\hat{\psi}_{\mathbf{C}}(\mathbf{x}), \tag{2.39}$$

$$\hat{\mathcal{H}}_{\mathbf{I}} = \int d^3\mathbf{x}\hat{\psi}^\dagger_{\mathbf{I}}(\mathbf{x})\hat{\mathcal{H}}_{\text{sp}}\hat{\psi}_{\mathbf{I}}(\mathbf{x}) + \frac{g}{2}\int d^3\mathbf{x}\hat{\psi}^\dagger_{\mathbf{I}}(\mathbf{x})\hat{\psi}^\dagger_{\mathbf{I}}(\mathbf{x})\hat{\psi}_{\mathbf{I}}(\mathbf{x})\hat{\psi}_{\mathbf{I}}(\mathbf{x}), \tag{2.40}$$

$$\hat{\mathcal{H}}^{(1)}_{\mathbf{I},\mathbf{C}} = g\int d^3\mathbf{x}\hat{\psi}^\dagger_{\mathbf{I}}(\mathbf{x})\hat{\psi}^\dagger_{\mathbf{I}}(\mathbf{x})\hat{\psi}_{\mathbf{C}}(\mathbf{x})\hat{\psi}_{\mathbf{C}}(\mathbf{x}) + \text{h.c.}, \tag{2.41}$$

$$\begin{aligned}\hat{\mathcal{H}}^{(2)}_{\mathbf{I},\mathbf{C}} = g\int d^3\mathbf{x}\hat{\psi}^\dagger_{\mathbf{I}}(\mathbf{x})\hat{\psi}^\dagger_{\mathbf{I}}(\mathbf{x})\hat{\psi}_{\mathbf{C}}(\mathbf{x})\hat{\psi}_{\mathbf{C}}(\mathbf{x}) \\ + \frac{g}{2}\int d^3\mathbf{x}\hat{\psi}^\dagger_{\mathbf{I}}(\mathbf{x})\hat{\psi}^\dagger_{\mathbf{I}}(\mathbf{x})\hat{\psi}_{\mathbf{I}}(\mathbf{x})\hat{\psi}_{\mathbf{I}}(\mathbf{x}) + \text{h.c.},\end{aligned} \tag{2.42}$$

$$\hat{\mathcal{H}}^{(3)}_{\mathbf{I},\mathbf{C}} = g \int d^3\mathbf{x} \hat{\psi}^\dagger_{\mathbf{C}}(\mathbf{x}) \hat{\psi}^\dagger_{\mathbf{C}}(\mathbf{x}) \hat{\psi}_{\mathbf{C}}(\mathbf{x}) \hat{\psi}_{\mathbf{I},\mathbf{C}}(\mathbf{x}) + \text{h.c.}. \tag{2.43}$$

Where h.c. stands for the Hermitian conjugate.

2.5.4 Different C-field Cases

We can then simulate different physical regimes depending how we deal with the various inter-region cross-terms and hence implement the C-field theory. In general there are the following cases:

(i) **The Projected GPE (PGPE)**: A C-field approach in which, in fact in all C-field techniques, the GPE-like time-evolution is restricted to the coherent **C** region. As the name says, this is the GPE projected onto the C-field using a projector. However this describes a micro-canonical (closed) system and thus the cross terms in the above interaction Hamiltonian are neglected (there is no coupling). This technique is valid for high temperatures $T \sim T_c$.

(ii) **Truncated Wigner PGPE (TWPGPE)**: This case is applicable when there are modes of low-occupation in the C-field and noise terms must be added to simulate the vacuum fluctuations. These quantum fluctuations can be included as an approximation using stochastic sampling of a Wigner distribution. Here the incoherent **I** region is unoccupied and the cross terms can be neglected. Applicable when $T \ll T_c$.

(iii) **The Stochastic Projected GPE (SPGPE)**: This technique is useful when the system involves transfer of energy and particles between the **C** and **I** regions, i.e. when we have a grand-canonical (open) quantum system. Additional noise terms are included via the truncated Wigner distribution. Thus it is clear that we retain the cross-terms in the interaction Hamiltonian. This technique can be thought of a full description of the system and is valid for high temperatures $T \sim T_c$.

Because the Wigner function theory is so often referred to, it is worthwhile to consider some of the formalism in some depth. It is also a good idea to note that a system can start in a physical situation where the first is valid and then a later time transition to another situation where the second is valid and then so on.

2.6 Wigner Distribution Theory

As may be already known, the main techniques of C-field formalism above can be set up using a Wigner distribution theory. This section considers an introduction to this. Originally

the function was introduced as some kind of 'quasi-probability' by E. Wigner in 1932 in motivation of the problem that in the classical limit of large quantum numbers, the probability distribution for an eigenfunction did not approach some classically-reasonable probabilty distribution [16].

2.6.1 Introduction to Coherent States

One of the most important classes of states in quantum mechanics is the coherent state. For instance these are used to describe modes containing large numbers of bosons, and also to approximate the ground state of coherent light emitted by a laser, and also the ground state of atoms in a BEC. In the quantum realm we know that the state of a particle or system cannot be defined as a single point in phase space due to reasons pertaining to Heisenberg's uncertainty principle. Coherent states however come the closest to a classical state and are thus minimum uncertainty states, with maximal coherence.

In the case of the phase space of position and momentum, the uncertainty is shared equally between position and momentum coordinates. Interestingly coherent states are not the eigenstates of any particular Hamiltonian but instead are the eigenstates of the annihilation operator of the harmonic oscillator and hence satisfies the eigenproblem:

$$\hat{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle, \tag{2.44}$$

where we note that since the annihilation operator is not Hermitian the coherent states don't form a orthogonal basis and the eigenvalues are complex numbers: $\alpha = \alpha_r + i\alpha_i$. The properties of these coherent states make them a suitable basis to express the Wigner function.

2.6.2 Wigner Representation of a Single Mode

Consider a generalised system in which for just one bosonic mode described by a density operator $\hat{\rho}$, we have a symmetrically ordered characteristic function given as a trace (Tr) over a function of the density operator:

$$\chi_W(\lambda, \lambda^*) = \mathrm{Tr}\{\hat{\rho}e^{\lambda\hat{a}^\dagger - \lambda^*\hat{a}}\}, \tag{2.45}$$

here λ is a complex variable. The density operator contains both statistical and quantum mechanical information about the system and defined for any general state $|\alpha_0\rangle$ as:

$$\hat{\rho} = \left|\alpha_0\right\rangle\left\langle\alpha_0\right|. \tag{2.46}$$

It turns out that the Wigner function always exists for any density matrix (ref.*Quantum Noise* book). The Wigner function expresses the density operator in the coherent state basis and defined as the Fourier transform of the characteristic function of Eq.(2.45):

$$W(\alpha,\alpha^*)=\frac{1}{\pi^2}\int d^2\lambda\, e^{\lambda^*\alpha-\lambda\alpha^*}\chi_W(\lambda,\lambda^*). \tag{2.47}$$

As mentioned earlier the Wigner function is similar to a classical probability distribution but exactly the same, hence why it's called a quasi-probability density function. But it can have either negative or positive values. One can then find the Wigner function from the density operator by first considering the von Neumann differential equation for the density operator, and then subsequently mapping the terms of this equation onto a partial differential equation for the Wigner function. For instance if we consider Eq.(2.45) but instead the operator product $\hat{a}\hat{\rho}$ we have:

$$\mathrm{Tr}\{\hat{a}\hat{\rho}e^{\lambda\hat{a}^\dagger-\lambda^*\hat{a}}\}=\left(\frac{1}{2}\lambda-\frac{\partial}{\partial\lambda^*}\right)\chi_W(\lambda,\lambda^*)\,, \tag{2.48}$$

Then taking the Fourier transform of Eq.(2.48):

$$\frac{1}{\pi^2}\int d^2\lambda\, e^{\lambda^*\alpha-\lambda\alpha^*}\left(\frac{1}{2}\lambda-\frac{\partial}{\partial\lambda^*}\right)\chi_W(\lambda,\lambda^*)=\left(\alpha+\frac{1}{2}\frac{\partial}{\partial\alpha^*}\right)W(\alpha,\alpha^*). \tag{2.49}$$

Hence we see that there is a so-called correspondence mapping:

$$\hat{a}\hat{\rho}\leftrightarrow\left(\alpha+\frac{1}{2}\frac{\partial}{\partial\alpha^*}\right)W(\alpha,\alpha^*). \tag{2.50}$$

For mapping of the terms we have:

$$\hat{a}^\dagger\hat{\rho}\leftrightarrow\left(\alpha^*-\frac{1}{2}\frac{\partial}{\partial\alpha}\right)W(\alpha,\alpha^*), \tag{2.51}$$

$$\hat{\rho}\hat{a}\leftrightarrow\left(\alpha-\frac{1}{2}\frac{\partial}{\partial\alpha^*}\right)W(\alpha,\alpha^*), \tag{2.52}$$

$$\hat{\rho}\hat{a}^\dagger\leftrightarrow\left(\alpha^*+\frac{1}{2}\frac{\partial}{\partial\alpha}\right)W(\alpha,\alpha^*). \tag{2.53}$$

These correspondences can be used to turn a operator differential equation into a Wigner differential equation, which is a key step in obtaining a stochastic differential equation of motion. It is also worthy to note that even an equation of motion for the Wigner function, a type of Fokker-Planck equation, is unwieldy and it is possible, under proper circumstance,

to solve a stochastic differential equation (SDE) instead. This SDE can be used to derive the SPGPE using phase-space mapping.

2.6.3 Extension to the Quantum Field

Now using the results for the single-mode treatment in the previous subsection we move to generalisation to quantum field theory. Thus we now consider a system with M modes in the C-field region and introduce the definition:

$$\int d^2\boldsymbol{\alpha} \equiv \prod_{n\in\mathbf{C}} \int d^2\alpha_n. \tag{2.54}$$

Where $\boldsymbol{\alpha} \equiv (\alpha_0, \alpha_1, \alpha_2, ..., \alpha_M)^T$ is a mode amplitude vector and the two-dimensional integral over it is equal to the product of the integrals of the set of mode amplitudes. Then we write the multimode Wigner function:

$$W_{\mathbf{C}}(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*) = \frac{1}{\pi^{2M}} \int d^2\boldsymbol{\lambda}\, e^{\boldsymbol{\lambda}^\dagger\boldsymbol{\alpha} - \boldsymbol{\alpha}^\dagger\boldsymbol{\lambda}} \chi_W(\boldsymbol{\lambda}, \boldsymbol{\lambda}^*). \tag{2.55}$$

Which is clearly a generalisation of Eq. (2.47). As before χ_W is the characteristic function for the C-field density operator $\hat{\rho}_{\mathbf{C}}$, and we note that $\boldsymbol{\alpha}^\dagger = (\boldsymbol{\alpha}^*)^T$. For a 'C-number' C-field defined as:

$$\hat{\psi}_{\mathbf{C}}(\mathbf{x}) = \sum_{n\in\mathbf{C}} \alpha_n \phi_n(\mathbf{x}), \tag{2.56}$$

the Wigner distribution moment yields the operator density average:

$$\int d^2\boldsymbol{\alpha} |\hat{\psi}_{\mathbf{C}}(\mathbf{x})|^2 W_{\mathbf{C}}(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*) = \left\langle \frac{\hat{\psi}^\dagger_{\mathbf{C}}(\mathbf{x})\hat{\psi}_{\mathbf{C}}(\mathbf{x}) + \hat{\psi}_{\mathbf{C}}(\mathbf{x})\hat{\psi}^\dagger_{\mathbf{C}}(\mathbf{x})}{2} \right\rangle \tag{2.57}$$

$$= \langle \hat{\psi}^\dagger_{\mathbf{C}}(\mathbf{x})\hat{\psi}_{\mathbf{C}}(\mathbf{x}) \rangle + \frac{\delta_{\mathbf{C}}(\mathbf{x}, \mathbf{x})}{2}. \tag{2.58}$$

In the last equality Eq. (2.58) the second term is the contribution from the projector or also the commutator term. This term accounts for half a quantum of noise per mode in the quantum field theory. Since the there are a finite number of modes included in the C-field region the contribution of this noise term (which is due to vacuum fluctuations) to the stochastic average is well-defined. The *truncated Wigner function* method allows for this. [5]

Like the single-mode case, we have correspondence mapping between the operator product to the partial differential Wigner terms generalised to many modes. To show this, we

introduce a convenient notion called the projected functional derivatives:

$$\frac{\bar{\delta}}{\bar{\delta}\psi_{\mathbf{C}}(\mathbf{x})} \equiv \sum_{n\in\mathbf{C}} \phi_n^*(\mathbf{x})\frac{\partial}{\partial\alpha_n}, \tag{2.59}$$

and

$$\frac{\bar{\delta}}{\bar{\delta}\psi_{\mathbf{C}}^*(\mathbf{x})} \equiv \sum_{n\in\mathbf{C}} \phi_n(\mathbf{x})\frac{\partial}{\partial\alpha_n^*}. \tag{2.60}$$

Then following a similar argument as in the single-mode case, using these functional derivatives we have that the correspondence between the new operator product between the C-field-density operators and the Wigner function:

$$\hat{\psi}_{\mathbf{C}}(\mathbf{x})\hat{\rho}_{\mathbf{C}} \longleftrightarrow \left(\psi_{\mathbf{C}}(\mathbf{x}) + \frac{1}{2}\frac{\bar{\delta}}{\bar{\delta}\psi_{\mathbf{C}}^*(\mathbf{x})}\right) W_{\mathbf{C}}, \tag{2.61}$$

$$\hat{\psi}_{\mathbf{C}}^\dagger(\mathbf{x})\hat{\rho}_{\mathbf{C}} \longleftrightarrow \left(\psi_{\mathbf{C}}^*(\mathbf{x}) - \frac{1}{2}\frac{\bar{\delta}}{\bar{\delta}\psi_{\mathbf{C}}(\mathbf{x})}\right) W_{\mathbf{C}}, \tag{2.62}$$

$$\hat{\rho}_{\mathbf{C}}\hat{\psi}_{\mathbf{C}}(\mathbf{x}) \longleftrightarrow \left(\psi_{\mathbf{C}}(\mathbf{x}) - \frac{1}{2}\frac{\bar{\delta}}{\bar{\delta}\psi_{\mathbf{C}}^*(\mathbf{x})}\right) W_{\mathbf{C}}, \tag{2.63}$$

$$\hat{\rho}_{\mathbf{C}}\hat{\psi}_{\mathbf{C}}^\dagger(\mathbf{x}) \longleftrightarrow \left(\psi_{\mathbf{C}}^*(\mathbf{x}) + \frac{1}{2}\frac{\bar{\delta}}{\bar{\delta}\psi_{\mathbf{C}}(\mathbf{x})}\right) W_{\mathbf{C}}. \tag{2.64}$$

Again we use these correspondence mappings to turn an original differential equation for the density operator (i.e. a von Neumann equation of motion) into a PDE for the Wigner function, $W_{\mathbf{C}}$.

2.6.4 The Truncated Wigner Function

Next we reconsider the Hamiltonian of Eq. (2.25) but for the time-evolution of the operator $\hat{\psi}_{\mathbf{C}}$, i.e. $\hat{\mathcal{H}}_{\mathbf{C}}$, for which we can write the von Neumann equation for the evolution of the density operator $\hat{\rho}_{\mathbf{C}}$:

$$i\hbar\frac{\partial\hat{\rho}_{\mathbf{C}}(t)}{\partial t} = \left[\hat{\mathcal{H}}_{\mathbf{C}}, \hat{\rho}_{\mathbf{C}}(t)\right]. \tag{2.65}$$

Now as already mentioned above in subsection 2.6.2, the right hand side of the von Neumann equation, Eq. (2.65) involves operator product terms for instance products of the density operator as well as $\hat{\psi}_{\mathbf{C}}$, and these terms can be mapped out using the correspondences of Eqs.

(2.61-2.64). Doing so yields the PDE for the Wigner function $W_{\mathbf{C}}$:

$$\left.\frac{\partial W_{\mathbf{C}}}{\partial t}\right|_{\hat{\mathcal{H}}_{\mathbf{C}}} = \int d^3\mathbf{x}\Bigg(\frac{iu}{4\hbar}\frac{\bar{\delta}}{\bar{\delta}\psi_{\mathbf{C}}(\mathbf{x})\bar{\delta}\psi^*_{\mathbf{C}}(\mathbf{x})}\psi^*_{\mathbf{C}}(\mathbf{x})\frac{\bar{\delta}}{\bar{\delta}\psi^*_{\mathbf{C}}(\mathbf{x})} + \text{h.c.} + \frac{i}{\hbar}\frac{\bar{\delta}}{\bar{\delta}\psi_{\mathbf{C}}(\mathbf{x})}(\hat{\mathcal{H}}_{\text{sp}} + u[|\psi_{\mathbf{C}}(\mathbf{x})|^2 - \delta_{\mathbf{C}}(\mathbf{x},\mathbf{x})])\psi_{\mathbf{C}}(\mathbf{x}) + \text{h.c.}\Bigg)W_{\mathbf{C}}. \tag{2.66}$$

Undoubtedly this equation is very complicated and not easily soluble. But as it turns out if the first two terms are dropped then the resulting equation is a more manageable one, and this is called the truncated Wigner approximation (TWA). The TWA is equivalent to the requirement that the modes are highly occupied in the C-field. In this approximation this equation is of the form of a Fokker-Planck equation with no diffusion terms but with drift terms:

$$\left.\frac{\partial W_{\mathbf{C}}}{\partial t}\right|_{\hat{\mathcal{H}}_{\mathbf{C}}} \cong \int d^3\mathbf{x}\left(\frac{i}{\hbar}\frac{\bar{\delta}}{\bar{\delta}\psi_{\mathbf{C}}(\mathbf{x})}(\hat{\mathcal{H}}_{\text{sp}} + g[|\psi_{\mathbf{C}}(\mathbf{x})|^2 - \delta_{\mathbf{C}}(\mathbf{x},\mathbf{x})])\psi_{\mathbf{C}}(\mathbf{x}) + \text{h.c.}\right)W_{\mathbf{C}}. \tag{2.67}$$

Then unsurprisingly Eq. (2.67), also known as the Liouville equation, can be mapped to a stochastic differential equation (SDE) that describes the motion of a realisation of the field number $\psi_{\mathbf{C}}(\mathbf{x})$:

$$i\hbar\frac{\partial\psi_{\mathbf{C}}(\mathbf{x})}{\partial t} = \hat{\mathcal{P}}_{\mathbf{C}}\{(\hat{\mathcal{H}}_{\text{sp}} + g[|\psi_{\mathbf{C}}(\mathbf{x})|^2 - \delta_{\mathbf{C}}(\mathbf{x},\mathbf{x})])\psi_{\mathbf{C}}(\mathbf{x})\}. \tag{2.68}$$

Eq. (2.68) is the truncated Wigner projected GPE (TWPGPE). There is no explicit noise term in this equation, but it is associated with stochastic fluctuations and the stochastic behaviour stems from the sampling of the initial conditions from the initial Wigner function. This is what we need in order to obtain the SPGPE.

2.7 The Projected Gross-Pitaevskii Equation

The Projected GPE(or PGPE) is applicable to Bose-degenerate gases at high temperatures and where we have a high degree of occupancy of the C-field. As it can be seen, it appears as just the GPE projected onto the C-field via the projector:

$$i\hbar\frac{\partial\psi_{\mathbf{C}}(\mathbf{x})}{\partial t} = \hat{\mathcal{P}}_{\mathbf{C}}\{(\hat{\mathcal{H}}_{\text{sp}} + g|\psi_{\mathbf{C}}(\mathbf{x})|^2)\psi_{\mathbf{C}}(\mathbf{x})\}. \tag{2.69}$$

One may also notice that the PGPE (Eq. (2.69)) is just the TWPGPE (Eq. (2.68)) with the delta term $\delta_{\mathbf{C}}(\mathbf{x},\mathbf{x})$ equal to zero as in the classical limit [5]. The derivation of the PGPE can be found in [9] and is similar to that for the bare GPE but with the restriction to the **C** region

only with no coupling to the **I** region. That is, the PGPE is used to describe the classical region of a finite temperature bosonic system in the regime of microcanonical theory.

2.8 The Stochastic Projected Gross-Pitaevskii Equation

The Stochastic projected GPE (or SPGPE) is grand-canonical theory and hence includes the interactions between the C-field region **C** and the incoherent region **I**. It is a also truncated Wigner theory that is applicable to high temperatures, i.e. $0.5T_c < T \leq T_c$, and also to slightly above T_c. It an extension of the PGPE involving extra terms pertaining to stochastic and spontaneous processes like condensate growth and scattering, obtained with the application of the TWPGPE method. This theory characterised by the temperature T and the chemical potential μ which can be time-dependent. Hence we keep the interaction Hamiltonian terms in the von Neumann equation of motion of Eq. (2.65). The procedure to obtain the SPGPE involves the derivation of a master equation for the reduced system by eliminating the reservoir degrees of freedom. Once obtained this master equation is mapped to a FPE for the motion of a Wigner function representation via the subsequent application of the correspondence mappings, Eqs. (2.62−2.64). Next applying the TWA so to neglect the third-order terms we can arrive at an equivalent TWPGPE from which we can get an SDE problem.

2.8.1 The Master Equation and Details of the SPGPE

To study the SPGPE and understand more about it's form and function, it is useful to write the full system master equation for the open quantum systems theory and work from there. More details on the complete derivation and insights involved in the process can be found in [12], here we give some of the important results. In this context the von Neumann equation for the full density operator of the open quantum system is: [14][15]

$$\frac{\partial\hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{\mathcal{H}},\hat{\rho}] \tag{2.70}$$

$$= -\frac{i}{\hbar}[\hat{\mathcal{H}}_{\mathbf{I}}+\hat{\mathcal{H}}_{\mathbf{C}}+\hat{\mathcal{H}}_{\mathbf{I},\mathbf{C}},\hat{\rho}]. \tag{2.71}$$

Thus we see that the interaction terms are included in the Hamiltonian. As mentioned already we now describe the condensate density operator and in terms of it write the full master equation:

$$\frac{\partial\hat{\rho}_{\mathbf{C}}}{\partial t} = \left.\frac{\partial\hat{\rho}_{\mathbf{C}}}{\partial t}\right|_{\text{Ham}} + \left.\frac{\partial\hat{\rho}_{\mathbf{C}}}{\partial t}\right|_{\text{growth}} + \left.\frac{\partial\hat{\rho}_{\mathbf{C}}}{\partial t}\right|_{\text{scatter}}. \tag{2.72}$$

Where the condensate density operator is defined as the trace of $\hat{\rho}$ over the non-condensate region:

$$\hat{\rho}_{\mathbf{C}} = \mathrm{Tr}_{\mathbf{I}}(\hat{\rho}). \tag{2.73}$$

We note that the first term in Eq. (2.72) is just that of the von Neumann equation, Eq. (2.65) used to obtain the TWPGPE. The second term as labelled describes the growth due to exchange of particles between the two regions and the last term describes the scattering or energy damping processes between the condensate and non-condensate regions. As outlined above, the full master equation is mapped to an equivalent FPE involving the Wigner function and subsequently applying the TWA trick, we can convert to the appropriate SDE equation describing the motion of a grand canonical open system, i.e. finite temperature BEC. Doing this one reaches the SPGPE which in its complete general form is:

$$\begin{aligned}(S)d\psi_{\mathbf{C}}(\mathbf{x}) = \hat{\mathcal{P}}_{\mathbf{C}}\Bigg\{&-\frac{i}{\hbar}\hat{\mathcal{L}}_{\mathbf{C}}\psi_{\mathbf{C}}(\mathbf{x})dt + \frac{G(\mathbf{x})}{k_B T}(\mu - \hat{\mathcal{L}}_{\mathbf{C}})\psi_{\mathbf{C}}(\mathbf{x})dt + dW_G(\mathbf{x},t)\\ &+ \int d^3\mathbf{x}' M(\mathbf{x}-\mathbf{x}')\frac{i\hbar\nabla\cdot\mathbf{j}_{\mathbf{C}}(\mathbf{x}')}{k_B T}\psi_{\mathbf{C}}(\mathbf{x})dt + i\psi_{\mathbf{C}}(\mathbf{x})dW_M(\mathbf{x},t)\Bigg\}.\end{aligned} \tag{2.74}$$

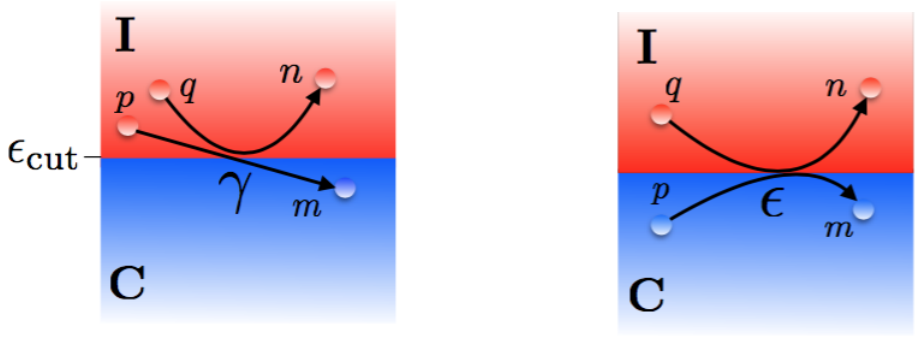
This is an SDE where the (S) denotes that this is a Strotonovich SDE. The first term in the the SPGPE is that of the PGPE for the microcanonical high temperature system. The second and third terms correspond to the condensate growth from the collisional interactions between the **C** region and the **I** region. Where the operator $\hat{\mathcal{L}}_{\mathbf{C}}$ is defined as:

$$\hat{\mathcal{L}}_{\mathbf{C}} \equiv (\hat{\mathcal{H}}_{\mathrm{sp}} + g|\psi_C(\mathbf{x},t)|^2)\hat{\psi}_{\mathbf{C}}(\mathbf{x}). \tag{2.75}$$

Also μ and T are the chemical potential and temperature of the thermal reservoir of region **I** particles. The fourth and last terms pertains to the number-conserving energy damping processes or scattering between the **C** and **I** regions. [5]

2.8.2 The Growth and Scattering Processes

Here we examine the important terms that appear in the SPGPE in some more detail. Considering first the growth terms, which are the second and third terms of Eq. (2.74) explain the condensate growth process from the scattering collisions between two non-condensate particles. Again μ and T characterise the thermal cloud or reservoir. Here $G(\mathbf{x})$ is the collision rate which is spatially dependent. Furthermore the complex and additive noise accompanying



(a) Growth (b) Scattering

Figure 2.4: Schematic showing the different types of interactions between the coherent and incoherent regions. In (a) two incoherent region particles collide leading to uneven energy transfer, causing one to enter the coherent region: this is the growth process. In (b) atoms from the coherent and incoherent regions collide but only transfer energy and remain in their respective regions: this is the scattering process.

the growth process $dW_G(\mathbf{x}', t)$ has the properties:

$$\langle dW_G^*(\mathbf{x}, t) dW_G(\mathbf{x}', t) \rangle = 2G(\mathbf{x})\delta_{\mathbf{C}}(\mathbf{x}, \mathbf{x}')dt, \tag{2.76}$$

and

$$\langle dW_G(\mathbf{x}, t) dW_G(\mathbf{x}', t) \rangle = \langle dW_G^*(\mathbf{x}, t) dW_G^*(\mathbf{x}', t) \rangle = 0. \tag{2.77}$$

As for the energy-damping terms also known as scattering terms, it is important to note that this scattering which involves collisions between the condensate and non-condensate regions that transfer only energy, and couples to the matterwave current density defined:

$$\mathbf{j}_{\mathbf{C}}(\mathbf{x}) \equiv \frac{i\hbar}{2m}\big(\nabla\psi_{\mathbf{C}}^*(\mathbf{x})\psi_{\mathbf{C}}(\mathbf{x}) - \psi_{\mathbf{C}}^*(\mathbf{x})\nabla\psi_{\mathbf{C}}(\mathbf{x})\big). \tag{2.78}$$

The scattering rate function $M(\mathbf{x} - \mathbf{x}')$ in the energy damping term of Eq. (2.74) is given by the Boltzmann integral. Associated with the scattering process, the real noise $dW_M(\mathbf{x}, t)$ is given by:

$$\langle dW_M(\mathbf{x}, t) dW_M(\mathbf{x}', t) \rangle = 2M(\mathbf{x} - \mathbf{x}')dt. \tag{2.79}$$

The scattering term has an integral over the scattering rate multiplied by the divergence of the condensate matterwave current density and also contains multiplicative noise, making it quite complicated than the term for the condensate growth. The two main types of inter-regional

interaction processes are further examined in Fig. (5.1a-5.1b). At this point we have now finally got the SPGPE and examined it's form and function. Thus we have described the BEC at finite temperatures. [12] [5]

Chapter 3

Stochastic Theory and the 1D SPGPE

As our work involves stochastic motion and theory, some overview of this topic especially as relating to the problem at hand is worthwhile. Here we'll consider the Langevin equation or stochastic differential equations and the different forms it can take. Later the 1D SPGPE since is a stochastic differential equation itself, it will follow the rules that we have outlined and hence allows us to recast it in a more numerically tractable equation.

3.1 The Langevin Equation

This is a conceptually familiar kind of motion which is the path of motion that a pollen grain or a speck of dust follows when suspended say in air or water. The small body gets randomly knocked about by the many particles of the medium and the cumulative effects of these numerous collisions leads to the stochastic motion we observe. Robert Brown studied this in detail in while looking at small particles in water-suspended pollen grains. Albert Einstein formulated the first theory of this type of motion. [11]

Another treatment of the same problem by Pierre Langevin, which is also simpler than Einstein's can be summerised here. The Langevin equation is a stochastic differential equation that is a generalisation of the Brownian motion to all stochastic motion and was first derived in the context of the problem of Brownian motion. We consider a particle of mass m moving in a viscous fluid such that the motion is damped, and also at the same time it is subject to a fluctuating random force due to the many medium particles. For a 1D system we have the equation of motion for the velocity:

$$\frac{dv}{dt} = -\beta v + \sqrt{f}\xi(t), \tag{3.1}$$

where $\beta = 6\pi a\eta/m$ is the drag coefficient with a being the radius of the particle, and η is

the viscosity of the fluid medium. The second term is the force due to the particle collisions where $\xi(t)$ corresponds to a noise process and f is a fluctuation factor. For different t and t' we have the properties that:

$$\langle \xi(t) \rangle = 0 \text{ and } \langle \xi(t)\xi(t') \rangle = \delta(t - t'). \tag{3.2}$$

The first says that the net effect of the collisions due to the fluid particles cancel. The second says that the noise processes for different times are independent processes. The general solution of the Langevin equation of Eq. (5.1) is given by:

$$v(t) = v(0)e^{-\beta t} + \sqrt{f} \int_0^t dt' e^{-\beta(t-t')} \xi(t'). \tag{3.3}$$

Which describes the velocity of the particle. Averaging the equation gives:

$$\langle v(t) \rangle = v(0)e^{-\beta t}, \tag{3.4}$$

Which also shows that in the long run the average velocity is zero, which makes sense. Furthermore the average of the square of the velocity is:

$$\langle v(t)^2 \rangle = v(0)^2 e^{-2\beta t} + \frac{f}{2\beta}(1 - e^{-2\beta t}), \tag{3.5}$$

With the variance in the velocity being:

$$\begin{aligned} \sigma_v(t) &= \langle v(t)^2 \rangle - \langle v(t) \rangle^2 && (3.6) \\ &= v(0)^2 e^{-2\beta t} + \frac{f}{2\beta}(1 - e^{-2\beta t}) - v(0)^2 e^{-2\beta t} && (3.7) \\ &= \frac{f}{2\beta}(1 - e^{-2\beta t}). && (3.8) \end{aligned}$$

As it turns out the motion of a soliton or the condensate itself at high temperatures through a broad thermal cloud of distinguishable species can be described by a similar Langevin equation. The condensate is subject to Brownian motion, what is known as a "quantum pollen grain". The thermal interactions between the coherent and incoherent region particles damps the motion of the condensate and also give rise to stochastic motion, as we have seen already in the previous chapter.

3.2 The Fluctuation-Dissipation Theorem

A particle undergoing Brownian motion, in the long run (i.e. in the limit as $t \to \infty$) comes to thermal equilibrium and the average energy per quadratic degree of freedom is then $k_{\mathrm{B}}T$ due to the equipartition theorem. So in this limit then Eq. (5.5) is:

$$\lim_{t\to\infty} \langle v(t)^2 \rangle = \frac{f}{2\beta}, \tag{3.9}$$

and shoving in $1/k_{\mathrm{B}}T$ in the expression for the mean energy per quandratic degree of freedom:

$$\lim_{t\to\infty} \left\langle \frac{1}{2} m v(t)^2 \right\rangle = \frac{fm}{4\beta} = \frac{1}{2} k_{\mathrm{B}} T, \tag{3.10}$$

which gives the fluctuation factor as:

$$f = \frac{2\beta k_{\mathrm{B}} T}{m} = 12\pi a \eta k_{\mathrm{B}} T. \tag{3.11}$$

This is the Fluctuation-Dissipation theorem, which says that energy is input to the Brownian particle by the random fluid particle motion and dissipated via the viscous drag. On the left hand side f corresponds to the fluctuations and on the right hand side we have dependence on temperature and viscosity, which correspond to the dissipation process. Furthermore in the long run, the rate of energy input due to the fluctuating forces on the particle is equal to the rate of energy lost. [11]

3.3 Representations of the Stochastic Differential Equation

The stochastic differential equation (SDE) is the generalisation of the description of the Brownian motion to any physical situation involving stochastic motion. An SDE is defined by the stochastic integral which can be defined in many ways. We consider the two main forms of the SDE, the Ito and Stratonovich SDEs. Both have their advantages and disadvantages, in particular the Ito SDE is mathematically more well-defined and the Ito increment is independent of any function of the time-dependent random variable that we consider. The Stratonovich representation is more physically useful and convenient also because the usual methods of calculus are applicable. However both arise naturally in studies involving stochastic motion and noise processes and one can quickly convert between one and another.

3.3.1 The Ito Representation of the SDE

In this representation the Ito stochastic integral is defined as: [18]

$$\int_{t_0}^{t} G(t)dW(t) = \lim_{n\to\infty}\left(\sum_{j=1}^{n} G(t_{j-1})\Delta W_j\right). \tag{3.12}$$

Where $\Delta W(t_j)$ is the Wiener process increments and $G(t)$ is some suitable function. From this integral the Ito SDE is defined, for a random variable $x(t)$ say, as:

$$x(t) = x(0) + \int_0^t a(x(t'), t')dt' + \int_0^t b(x(t'), t')dW(t'). \tag{3.13}$$

Then a random variable $x(t)$ satsifies an Ito SDE if we have:

$$(\mathrm{I})dx(t) = a(x(t), t)dt + b(x(t), t)dW(t). \tag{3.14}$$

The Wiener process is a real stochastic process and a Gaussian quantity defined as the integral of the fluctuation process $\xi(t)$, as in Eq. (5.1):

$$W(t) = \int_0^t dt'\xi(t'). \tag{3.15}$$

With further properties, which can be calculated from Eq. (4.15), given by:

$$\langle W(t)\rangle = 0 \ \text{ and } \ \langle W(t)^2\rangle = t. \tag{3.16}$$

And the same rule holds for the differential Wiener increment $dW(t)$. For the development of the formulation further, it should be noted that inside integrals it is convenient to use what are called Ito rules:

$$dW(t)^2 = dt, \tag{3.17}$$

$$dW(t)dt = 0, \tag{3.18}$$

$$dt^{N+1} = dW(t)^{N+2} = 0 \quad \text{, for} \quad N \geq 1. \tag{3.19}$$

It is also worthwhile to note that in the Ito representation the Wiener process increment $dW(t)$, as in Eq. (4.14) does not depend on anything that has happened up to time t. This means that $dW(t)$ is independent of $x(t)$ and any function of $x(t)$, which makes the Ito representation convenient. The Ito formulation is mathematically less technical and useful for technical manipulation and is thus arises often for this reason. [11]

3.3.2 The Stratonovich Representation of the SDE

Similarly in this alternate representation the stochastic integral is defined by: [18]

$$\int_{t_0}^{t} G(t)dW(t) = \lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{G(t_{j-1})+G(t_j)}{2}\Delta W_j\right). \tag{3.20}$$

Where we note the new term in the sum and the factor of $1/2$. From which we get the Stratonovich definition of the SDE as:

$$dx(t) = a(x(t),t)dt + \frac{b(x(t))+b(x(t+dt))}{2}dW(t). \tag{3.21}$$

With further manipulation, see (*ref. x*), we can get this in the form:

$$(\mathrm{S})dx(t) = \left(a\big(x(t),t\big) + \frac{1}{2}b\big(x(t),t\big)\frac{\partial b\big(x(t),t\big)}{\partial x}\right)dt + b\big(x(t),t\big)dW(t). \tag{3.22}$$

The good thing about the Stratonovich representation is that it utilises ordinary calculus methods, for instance for a function $f(x)$ the corresponding SDE is:

$$(\mathrm{S})dx(t) = \frac{df\big(x(t)\big)}{dx}\Big(a\big(x(t),t\big)dt + b\big(x(t),t\big)dW(t)\Big), \tag{3.23}$$

which can get using calculus: $df\big(x(t)\big) = f'\big(x(t)\big)dx$ and just substituting $(S)dx(t) = a\big(x(t),t\big)dt + b\big(x(t),t\big)dW(t)$. The Stratonovich representation for this reason is very useful and it turns out that this representation describes physical situations and problems better, i.e. the physics favours the Stratonovich form more than the Ito form. However as can be seen from Eq. (4.20), the Wiener increment $dW(t)$ is not always "in the future" of the integrand and so it is not independent of $x(t)$ or any function of it, instead it is correlated with it.

As we have seen one can switch from one representation to the other depending if it is convenient for the reasons given above: [11]

Stratonovich		Ito
$a(x),\ \ b(x)$	$\rightarrow$	$a(x)+\frac{1}{2}b(x)\frac{\partial b}{\partial x},\ \ b(x)$
$a(x)-\frac{1}{2}b(x)\frac{\partial b}{\partial x},\ \ b(x)$	$\leftarrow$	$a(x),\ \ b(x)$

3.4 The Stratonovich and Ito forms of the 1D SPGPE

For our intentions, for the sake of numerical tractability as already mentioned in chapter 2 and in the last section, the dynamics of the BEC can be described by an equivalent SDE or

a Langevin equation, which we just saw can take the form of either the Stratonovich or Ito SDE. Here we first write the full 3D SPGPE in the Stratonovich as it is customary to do so:

$$(\mathrm{S})d\psi(\mathbf{x},t) = d\psi(\mathbf{x},t)|_{\mathcal{H}} + d\psi(\mathbf{x},t)|_{\gamma} + (\mathrm{S})d\psi(\mathbf{x},t)|_{\epsilon}, \tag{3.24}$$

where we note that:

$$d\psi(\mathbf{x},t)|_{\mathcal{H}} = \hat{\mathcal{P}}\{\hat{\mathcal{L}}\psi(\mathbf{x},t)dt\}, \tag{3.25}$$

$$i\hbar d\psi(\mathbf{x},t)|_{\gamma} = \hat{\mathcal{P}}\{-i\gamma\hat{\mathcal{L}}\psi(\mathbf{x},t)dt + i\hbar dW(\mathbf{x},t)\}, \tag{3.26}$$

$$(\mathrm{S})i\hbar d\psi(\mathbf{x},t)|_{\epsilon} = \hat{\mathcal{P}}\{V_{\epsilon}(\mathbf{x},t)\psi(\mathbf{x},t)dt - \hbar\psi(\mathbf{x},t)dU(\mathbf{x},t)\}, \tag{3.27}$$

and we further have:

$$\hat{\mathcal{L}} = \hat{\mathcal{H}} + g|\psi(\mathbf{x},t)|^2, \quad \text{and} \quad \hat{\mathcal{H}} = -\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{ext}}(\mathbf{x}). \tag{3.28}$$

Also $\hat{\mathcal{H}}$ is single-particle Hamiltonian, $V_{\epsilon}(\mathbf{x},t)$ is the energy damping potential whose form we shall express a little later. Eq. (4.30) corresponds to the PGPE, Eq. (4.31) corresponds to the number damping or growth process and Eq. (4.35) corresponds to the energy damping or scattering process. Note the projection operators, these impose the high energy cut-off in the basis of single particle states $\phi_n(\mathbf{x})$ satisfying $\hat{\mathcal{H}}\phi_n(\mathbf{x}) = \epsilon_n\phi_n(\mathbf{x})$. Again for our purposes the number damping term can be dropped as in the situation of a two-component system where both components are distinct species then by consequence of the conservation of spin in collisions between the two components, number damping is not possible (and doesn't happen). But for our purposes we are considering effective 1D dynamics as already talked about in chapter 1, hence the resulting energy damped 1D SPGPE is then: [7]

$$(\mathrm{S})i\hbar d\psi(x,t) = \hat{\mathcal{P}}\{([\hat{\mathcal{L}} + V_{\epsilon}(x,t)]dt - \hbar dU(x,t))\psi(x,t)\}. \tag{3.29}$$

We note that $\hat{\mathcal{L}}$ is reduced to the 1D partial differential operator with now $g = g_{1\mathrm{D}} = 2\hbar\omega_{\perp,1}a_1$, and a_1 is the s-wave scattering length of component 1 (the condensate). Where we also have the explicit form of the 1D energy damping potential:

$$V_{\epsilon}(x,t) \equiv -\hbar\int dx'\epsilon(x-x')\frac{\partial}{\partial x'}j(x',t), \tag{3.30}$$

where $j(x',t)$ is the matterwave current density the form of which is given by: $j(x',t) \equiv$

$(i\hbar/2m)[(\partial_x\psi^*)\psi - \psi^*(\partial_x\psi)]$, and we got:

$$\epsilon(x) \equiv \frac{a_{12}^2}{e^{|\mu|/k_{\rm B}T} - 1} \int_{-\infty}^{\infty} \frac{dk\, e^{ikx}}{\left(2\pi(a_{\perp,1})^2\right)^{1/2}} \text{erfcx}\left(\frac{a_{\perp,1}|k|}{\sqrt{2}}\right). \tag{3.31}$$

Here a_{12} is the inter-component s-wave scattering length, μ and T are the chemical potential and temperature of component 2 (thermal reservoir) respectively, and $\text{erfcx}(x) \equiv e^{x^2}\text{erfc}(x)$ is the scaled complementary error function. We further note that the noise that appears $dU(x,t)$, is a real Wiener process with the correlator:

$$\langle dU(x,t)dU(x',t)\rangle = \frac{2k_{\rm B}T}{\hbar}\epsilon(x-x')dt. \tag{3.32}$$

Then we can convert the 1D Stratonovich SPGPE, Eq. (4.37) to Ito form to get:

$$(\text{I})i\hbar d\psi(x,t) = \hat{\mathcal{P}}\{(\hat{L}dt - \hbar dU(x,t))\psi(x,t)\}, \tag{3.33}$$

where $\hat{L}\psi(x,t) = \left(\hat{\mathcal{L}} + V_\epsilon(x,t) - ik_{\rm B}T\epsilon(0)\right)\psi(x,t)$, and we have defined $\epsilon(0)\psi(x,t)$ as:

$$\epsilon(0)\psi(x,t) \equiv \int dx'\epsilon(x-x')\delta(x,x')\psi(x',t). \tag{3.34}$$

This is to account for the Stratonovich correction. Thus we have seen that the full 3D SPGPE, in its most general form given by Eq. (2.74), can be reduced to an effective 1D equation and expressed in the either the Stratonovich and Ito SDEs, depending on whichever one is more convenient at the time. These forms also lead to numerical tractability as well, particularly well-suited for the physical problem that is considered, in this case, the motion of the Kohn modes in the vicinity of the thermal reservoir.

Chapter 4

Analytical Results

Now as part of the problem of examining the motion of the Kohn mode of the condensate, we use all the information we have outlined in the previous chapters and seek to obtain the equation of motion and other parameters relating to the dynamics. Since the Kohn mode is the center of mass oscillation of the condensate density, that is the ground state density, we ought to use a well-defined solution of the GPE. In this case we start with the Thomas-Fermi wavefunction which is a very good approximation to the actual groundstate except at the edges away from the peak density as we have already seen in chapter 2. So it is natural to use this anzats and work from there, since there is no other appropriate and well-defined anzats and one cannot write an explicit and tractable equation for the true groundstate wavefunction.

4.1 Preliminaries

As stated earlier the Thomas-Fermi wavefunction is given by:

$$\psi_{\mathrm{TF}}(x,t) = \sqrt{\frac{\mu}{g_1}\left(1 - \frac{(x - X(t))^2}{R^2}\right)}\, e^{iP(t)x/\hbar}, \tag{4.1}$$

The density is then:

$$n_{\mathrm{TF}}(x,t) \equiv |\psi_{\mathrm{TF}}(x,t)|^2 = n_0\left(1 - \frac{(x - X(t))^2}{R^2}\right), \tag{4.2}$$

where also $X(t)$ is the centre of mass position of the condensate density and also:

$$n_0 = \frac{\mu}{g_1}, \quad R \equiv \sqrt{\frac{2\mu}{m\omega_x^2}}, \quad \text{and} \quad g_1 \equiv 2\hbar\omega_{\perp,1}a_1. \tag{4.3}$$

Again g_1 is the 1D interaction strength of the condensate (i.e. the coherent region or $|1\rangle$) and $\omega_{\perp,1}$ is the transverse trapping frequency. We have then, where $M = N_1 m$ for m being the mass of a condensate atom:

$$X(t) = X_0 \cos\omega t + \frac{P_0}{M\omega}\sin\omega t, \quad X(0) = X_0, \tag{4.4}$$

$$\dot{X}(t) = -X_0\omega \sin\omega t + \frac{P_0}{M}\cos\omega t, \quad \dot{X}(0) = V_0 = \frac{P_0}{M}, \tag{4.5}$$

The total collective momentum of the density is given by $P(t)$:

$$P(t) = P_0 \cos\omega t - M\omega X_0 \sin\omega t, \quad P(0) = P_0, \tag{4.6}$$

$$\dot{P}(t) = -P_0\omega \sin\omega t - M\omega^2 X_0 \cos\omega t, \quad \dot{P}(0) = -M\omega^2 X_0. \tag{4.7}$$

Then we have that the thermal cloud (i.e. the incoherent region or $|2\rangle$), which is described by the chemical potential μ, temperature T and a high energy cutoff $\epsilon_{\mathrm{cut},2}$, interacts with the coherent region by means of an effective potential that couples to the gradient of the matterwave current density $j(x,t)$. We can further take $\epsilon_{\mathrm{cut},2} \cong 0$, this is because we are in the regime where the T is much higher than $T_{c,2}$ so the thermal cloud is entirely non-condensed. The effective or energy damping potential, given by Eq. (3.30) damps the energy of the condensate and opposes its motion, thus slowing it down. Taking this potential we carry out integration by parts:

$$\begin{aligned} V_\epsilon(x,t) &= -\hbar \int \mathrm{d}x'\, \epsilon(x-x') \frac{\partial}{\partial x'} j(x',t) && (4.8)\\ &= -\hbar \left(\epsilon(x-x') j(x',t)|_{X-R}^{X+R} - \int \mathrm{d}x' \, \frac{\partial}{\partial x'} \epsilon(x-x') j(x',t) \right) && (4.9)\\ &= \hbar \int \mathrm{d}x' \, \frac{\partial}{\partial x'} \epsilon(x-x') j(x',t), && (4.10) \end{aligned}$$

And we get

$$\begin{aligned} \frac{\partial}{\partial x'}\epsilon(x-x') &= \mathcal{M} \int_{-\infty}^{\infty} dk \, \frac{\partial}{\partial x'} e^{ik(x-x')} f(|k|) && (4.11)\\ &= \mathcal{M} \int_{-\infty}^{\infty} dk \, (-ik) e^{ik(x-x')} f(|k|). && (4.12) \end{aligned}$$

Where

$$\mathcal{M} = \frac{a_{12}^2}{e^{|\mu|/k_{\rm B}T} - 1}, \tag{4.13}$$

and

$$f(|k|) = \frac{1}{(2\pi(a_{\perp,1})^2)^{1/2}} \text{erfcx}\left(\frac{a_{\perp,1}|k|}{\sqrt{2}}\right). \tag{4.14}$$

As for the current density we calculate:

$$j(x',t) = \frac{n_{\rm TF}(x',t)\hbar}{M}\frac{\partial\theta(x',t)}{\partial x'} = \frac{n_{\rm TF}(x',t)P(t)}{M} = n_{\rm TF}(x',t)\dot{X}(t) \tag{4.15}$$

for

$$\theta = \frac{P(t)}{\hbar}x, \quad P(t) = M\dot{X}(t). \tag{4.16}$$

Next we find the equation for the center of mass coordinate of a system of bosons governed by the Ito 1D SPGPE, Eq. (3.33). Carrying on we need to obtain a Langevin equation for the field momentum:

$$P(\psi,\psi^*) = \int dx\psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x,t), \tag{4.17}$$

through the change of variables using Ito rules:

$$\begin{aligned} dP(\psi,\psi^*) = \int dx \Bigg[& \frac{\bar{\delta}P(\psi,\psi^*)}{\bar{\delta}\psi(x)}d\psi(x) \\ & + \frac{k_{\rm B}T}{\hbar}\int dx'\left(\frac{\bar{\delta}^2P(\psi,\psi^*)}{\bar{\delta}\psi(x)\bar{\delta}\psi^*(x')}\psi^*(x) - \frac{\bar{\delta}^2P(\psi,\psi^*)}{\bar{\delta}\psi(x)\bar{\delta}\psi(x')}\psi(x)\right)\psi(x')\epsilon(x-x')dt\Bigg]. \end{aligned} \tag{4.18}$$

When the functional derivatives of $P(\psi,\psi^*)$ are calculated taking into account the Stratonovich correction, the results we arrive at are:

$$dP(t) = F(t)dt + \sqrt{G(t)}dW(t), \tag{4.19}$$

$$F(t) = -\int dx\; n_{\rm TF}(x,t)\frac{\partial}{\partial x}V_{\rm ext}(x,t) + F_\epsilon(x,t). \tag{4.20}$$

$$G(t) = 2\hbar k_{\rm B}T\int dx\,dx'\epsilon(x-x')\frac{\partial}{\partial x}n_{\rm TF}(x,t)\frac{\partial}{\partial x'}n_{\rm TF}(x',t). \tag{4.21}$$

We then evaluate:

$$\frac{\partial}{\partial x}V_\epsilon(x,t) = \hbar\frac{\partial}{\partial x}\int dx'\,\frac{\partial}{\partial x'}\epsilon(x-x')j(x',t) \tag{4.22}$$

$$= \mathcal{M}\hbar\int dx'\int_{-\infty}^{\infty} dk\,k^2 e^{ik(x-x')}f(|k|)n_{\rm TF}(x',t)\dot{X}(t), \tag{4.23}$$

which is used to evaluate the effective or energy damping force $F_\epsilon(x,t)$, where $f = f(|k|)$:

$$F_\epsilon(x,t) = -\int dx\,n_{\rm TF}(x,t)\frac{\partial}{\partial x}V_\epsilon(x,t) \tag{4.24}$$

$$= -\mathcal{M}\hbar\dot{X}(t)\int_{-\infty}^{\infty} dk\,k^2 f\frac{\mu^2R^2}{g^2}\left(\frac{4\sin(kR)}{(kR)^3}-\frac{4\cos(kR)}{(kR)^2}\right)^2 \tag{4.25}$$

$$= -\Gamma\dot{X}(t). \tag{4.26}$$

Which we get if we substitute Eq. (5.2) and evaluate using the limits $X \pm R$, and we have:

$$\Gamma \equiv \mathcal{M}\hbar\int_{-\infty}^{\infty} dk\,k^2 f(|k|)\left(\frac{\mu^2R^2}{g^2}\right)\left(\frac{4\sin(kR)}{(kR)^3}-\frac{4\cos(kR)}{(kR)^2}\right)^2. \tag{4.27}$$

Then the other term:

$$-\int dx\,n_{\rm TF}(x,t)\frac{\partial V_{\rm ext}(x,t)}{\partial x} = -\int_{X-R}^{X+R}\frac{dx\mu}{g}\left(1-\left(\frac{x-X(t)}{R}\right)^2\right)m\omega^2 x \tag{4.28}$$

$$= -\frac{4\mu m\omega^2 R}{3g}X(t) \tag{4.29}$$

$$= -m\omega^2 N_1 X(t). \tag{4.30}$$

Where we have N_1 number of atoms in the condensate, and $G(t)$:

$$N_1 = \int_{-\infty}^{\infty} n_{\rm TF}(x,t)dx = \frac{4\mu R}{3g}. \tag{4.31}$$

$$G(t) = 2\hbar k_{\rm B}T\int dx\int {\rm d}x'\,\epsilon(x-x')\frac{\partial}{\partial x}n_{\rm TF}(x,t)\frac{\partial}{\partial x'}n_{\rm TF}(x',t) \tag{4.32}$$

$$= 2\hbar k_{\rm B}T\int dx\,n_{\rm TF}(x,t)\frac{\partial}{\partial x}\int dx'\,\frac{\partial}{\partial x'}\epsilon(x-x')n_{\rm TF}(x',t) \tag{4.33}$$

$$= 2\hbar k_{\rm B}T\mathcal{M}\int_{-\infty}^{\infty} dk\,k^2 f\int_{X-R}^{X+R} dx\,n_{\rm TF}e^{ikx}\int_{X-R}^{X+R} dx'\,n_{\rm TF}e^{-ikx'} \tag{4.34}$$

$$= 2k_{\rm B}T\Gamma. \tag{4.35}$$

Then we get the SDE for the field momentum:

$$dP(t) = -\left(\Gamma\dot{X}(t) + m\omega^2 N_1 X(t)\right)dt + \sqrt{2k_{\rm B}T\Gamma}dW(t). \tag{4.36}$$

Where also $dW(t)$ is just a Wiener process increment and is a real quantity with $\langle dW(t)\rangle = 0$, and $\langle dW(t)^2\rangle = dt$. Then we can find the condensate center of mass velocity:

$$V(t) \equiv \dot{X}(t) = \frac{P(t)}{N_1 m}, \tag{4.37}$$

From which we can quickly write the SDE for the condensate velocity:

$$\begin{aligned} dV(t) &= \frac{dP(t)}{N_1 m} && (4.38)\\ &= -\frac{1}{N_1 m}\left[\left(\Gamma\dot{X}(t) + m\omega^2 N_1 X(t)\right)dt - \sqrt{2k_{\rm B}T\Gamma}dW(t)\right] && (4.39)\\ &= -(\zeta V(t) + \omega^2 X(t))dt + \sqrt{2k_{\rm B}\tau\zeta}dW(t). && (4.40)\end{aligned}$$

Thus the crucial quantities are found to be:

$$dV(t) = -(\zeta V(t) + \omega^2 X(t))dt + \sqrt{2D}dW(t). \tag{4.41}$$

$$\zeta \equiv \frac{\Gamma}{N_1 m} = \frac{9\hbar N_1 a_{12}^2}{16mR^3(2\pi(a_{\perp,1})^2)^{1/2}}\,\frac{\mathcal{I}}{e^{|\mu|/k_{\rm B}T}-1}, \tag{4.42}$$

$$\mathcal{I} \equiv \int_{-\infty}^{\infty} dz\, z^2\, \text{erfcx}\left(\frac{\lambda|z|}{\sqrt{2}}\right)\left(\frac{4\sin(z)}{z^3} - \frac{4\cos(z)}{z^2}\right)^2. \tag{4.43}$$

We note that Eq. (4.42) is the damping or decay rate, which is likely written here in its most fundamental form. Eq. (4.43) is a geometric factor which is found from Eq. (4.27) by letting $z = kR$ and $\lambda = a_{\perp,1}/R$. Furthermore we note also that:

$$D = k_{\rm B}\tau\zeta = \frac{k_{\rm B}T\zeta}{N_1 m}, \tag{4.44}$$

is the condensate groundstate velocity diffusion coefficient, which resembles the classical Brownian type diffusion coefficient, and also satifies the fluctuation-dissipation theorem. Now we have turned the stochastic equation of motion for the Bose field in the Wigner representation to an Ornstein-Uhlenbeck equation for the condensate center of mass velocity.

Then if one works through the details starting with Eq. (4.41) and going about solving any

SDE, then it is found that for the condensate groundstate with initial velocity of $V(0) = V_0$, the subsequent time evolution is given by the solution:

$$V(t) = V_0\, e^{-\zeta t}\cos(\omega_1 t) + \sqrt{2D}\int_0^t e^{-\zeta(t-t')/2} dW(t'). \tag{4.45}$$

Where $\omega_1 = \sqrt{\omega^2 - \zeta^2}$, is the modified frequency which is smaller than the undamped frequency. Thus it can be clearly seen that we have a damped simple harmonic motion superimposed with a noisy random motion. It is also worthy of noting that here we consider the underdamped regime i.e. where $\zeta < \omega$. Then the mean velocity is $\langle V(t)\rangle = V_0\, e^{-\zeta t}\cos(\omega_1 t)$ since we already know that $\langle dW(t)\rangle = 0$. The variance of the velocity-squared is:

$$\langle V(t)^2\rangle = V_0^2\, e^{-2\zeta t}\cos^2(\omega_1 t) + \frac{2D}{\zeta}(1 - e^{-2\zeta t}). \tag{4.46}$$

The solution of Eq. (4.45) is the solution of the homogeneous non-driven oscillator plus an additive noise term. Clearly then the steady-state variance is given by:

$$\lim_{t\to\infty}\langle V(t)^2\rangle = \frac{2D}{\zeta} = \frac{2k_{\rm B}T}{N_1 m}. \tag{4.47}$$

Furthermore note that Eq.(4.43) can also be expressed into the form:

$$\Gamma = \bar{\Gamma}\int_0^\infty dz\, z^2\, \text{erfcx}\left(\frac{\lambda|z|}{\sqrt{2}}\right)\left(\frac{4\sin(z)}{z^3} - \frac{4\cos(z)}{z^2}\right)^2. \tag{4.48}$$

$$= \bar{\Gamma}\, I(\lambda). \tag{4.49}$$

Where:

$$\bar{\Gamma} = \frac{2\mathcal{M}\hbar\mu^2}{g^2 R(2\pi(a_{\perp,1})^2)^{1/2}}. \tag{4.50}$$

Then calculating again:

$$\bar{\Gamma} = \frac{2\mathcal{M}\hbar\mu^2}{g^2 R(2\pi(a_{\perp,1})^2)^{1/2}}, \tag{4.51}$$

$$= \frac{2\mathcal{M}}{\sqrt{2\pi}}\frac{\hbar\mu}{g}\frac{3N_1}{4R}\frac{1}{Ra_\perp} \tag{4.52}$$

$$= \frac{\mathcal{M}}{a_\perp a_s}\frac{3N_1}{8\sqrt{2\pi}}\frac{\omega_x}{\omega_\perp}(\hbar\omega_x)\frac{m}{\hbar} \tag{4.53}$$

$$= \frac{3N_1 m}{8\sqrt{2\pi}\hbar}\frac{\mathcal{M}}{a_s a_\perp}\frac{\omega_x}{\omega_\perp}\hbar\omega_x \tag{4.54}$$

Where from Eq. (4.51) to (4.52) used the expression $\mu/g = 3N_1/4R$ and from Eq. (4.52) to

(4.53) used $\mu/R^2 = m\omega_x^2/2$. Then from Eq. (4.49) way above, $\Gamma = \bar{\Gamma}\, I(\lambda)$ we have:

$$\zeta = \frac{\Gamma}{N_1 m} = \frac{\bar{\Gamma}}{N_1 m} I(\lambda) \tag{4.55}$$

$$= \bar{\zeta}\, I(\lambda) \tag{4.56}$$

$$= \tilde{\zeta}\zeta_0\, I(\lambda) \tag{4.57}$$

$$= \zeta'\zeta_0 \tag{4.58}$$

Here the new dimensionless damping rate is:

$$\zeta' = \tilde{\zeta}\, I(\lambda), \quad \tilde{\zeta} = \frac{3}{8\sqrt{2\pi}} \frac{\mathcal{M}}{a_s a_\perp} \frac{\omega_x}{\omega_\perp}, \quad \text{and} \quad \zeta_0 = 1/t_0 = \omega_x. \tag{4.59}$$

Also $I(\lambda)$ is the integral way above. For the values of $\mathcal{M} = 2.0 \times 10^{-4}$, using a z-grid, with $\lambda = 0.006667$, for $a_s = 1.5a_0$, where a_0 is the Bohr radius, $a_{\perp,1} = 0.77 \times 10^{-6}$m, $\omega_x = 5.4777$ Hz, and $\omega_{\perp,1} = 1.232 \times 10^3$ Hz, $I(\lambda) = 8.1363$, and the value of the dimensionless analytical decay rate is found to be $\zeta' = 0.010496$. As seen in the first chapter we consider the case where $a_{\perp,1} \ll a_x \ll R$. The integrand of $I(\lambda)$ is shown in Fig (x). This integral

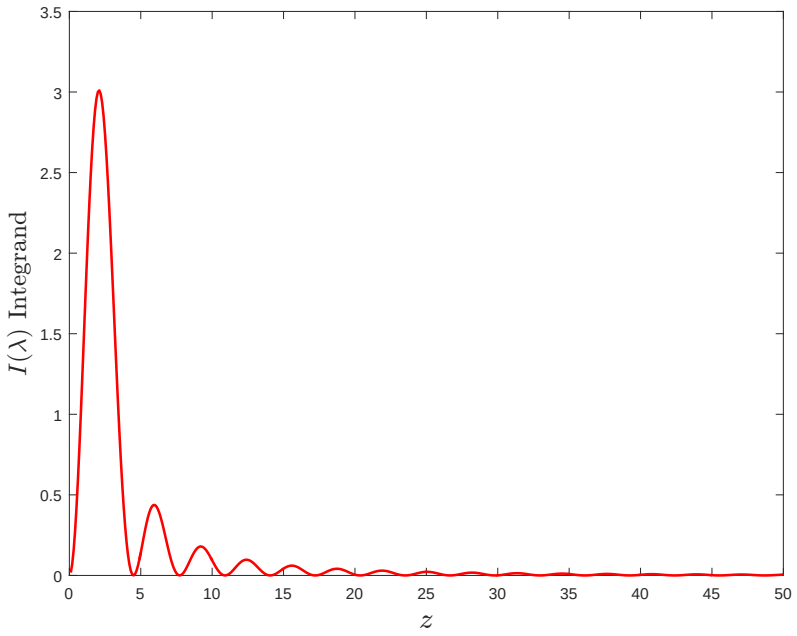


Figure 4.1: The integrand function of $I(\lambda)$ plotted for $z > 0$. The integrand is even and converges.

serves as a geometric factor, and is evaluated using the Simpson's rule for integration. In the next chapter we will compare these results with the numerics.

Chapter 5

Numerical Results

In the next section we consider the results obtained from running the energy damped 1D SPGPE of Eq. (3.29), and validate the results of the analytical section as well as learning about the energy damping effects of the Kohn modes. Since the Kohn modes are center of mass oscillations of the condensate density, we analyse sinusoidal oscillations and expect thermalisation, noisy and spontaneous processes appearing on the modes. It should be noted that there are bound to be differences and inconsistencies to expect when comparing the numerics and the analytics of the previous section since there is no exact solution for the condensate gound state, while the SPGPE utilises the true ground state.

5.1 Kohn Mode Damping

In solving and integrating the SPGPE, we have the following parameters in dimensionless units, that are expressed in the harmonic oscillator units that should be defined on the get go:

$$\tilde{R} = \frac{R_x}{a_x} = \sqrt{2\tilde{\mu}} \tag{5.1}$$

And the particle number is:

$$N = \frac{4}{3}\mu^{3/2}\left(\frac{a_x}{a_s}\right)\left(\frac{1}{2\hbar\omega_x(\hbar\omega_\perp)^2}\right)^{1/2} \tag{5.2}$$

And:

$$\tilde{N} = \frac{4}{3}\frac{\tilde{\mu}^{3/2}}{\tilde{g}}2^{1/2}. \tag{5.3}$$

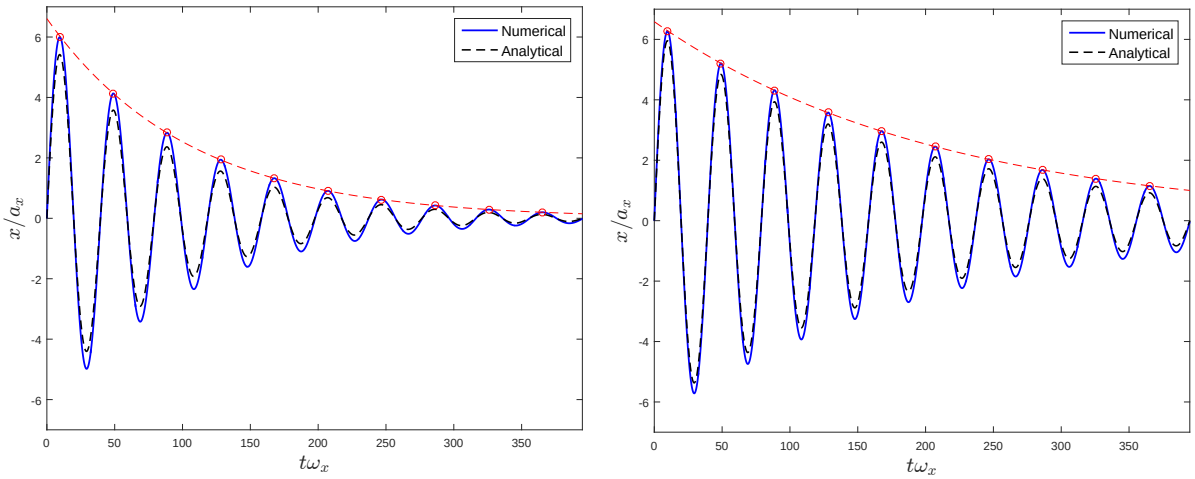
Where the tilde are dimensionless. Where also:

$$\tilde{g} = \frac{2\hbar\omega_\perp a_s}{\hbar\omega_x a_x} \quad \text{and} \quad \tilde{\mu} = \frac{\mu}{\hbar\omega_x}. \tag{5.4}$$

These were also used in the analytical section to calculate the damping rate ζ. Comparing the analytical decay rate with the numerical rate at $T = 0$ we get, for values of $\mathcal{M} = 0.0002$, $g = 0.00309$ the plots shown in Figure (x). Similarly for $T = 0$ and $\mathcal{M} = 0.0001$, $g = 0.00309$ we have Fig. (x). As it can be seen the decaying oscillations are given by Fig. (4.45) with out any additive noise, and thus the position is:

$$X(t) = X_0\, e^{-\zeta t}\sin(\omega_1 t). \tag{5.5}$$

For the case of $\mathcal{M} = 0.0002$ as in Fig. (5.1a), the analytical decay rate was calculated to

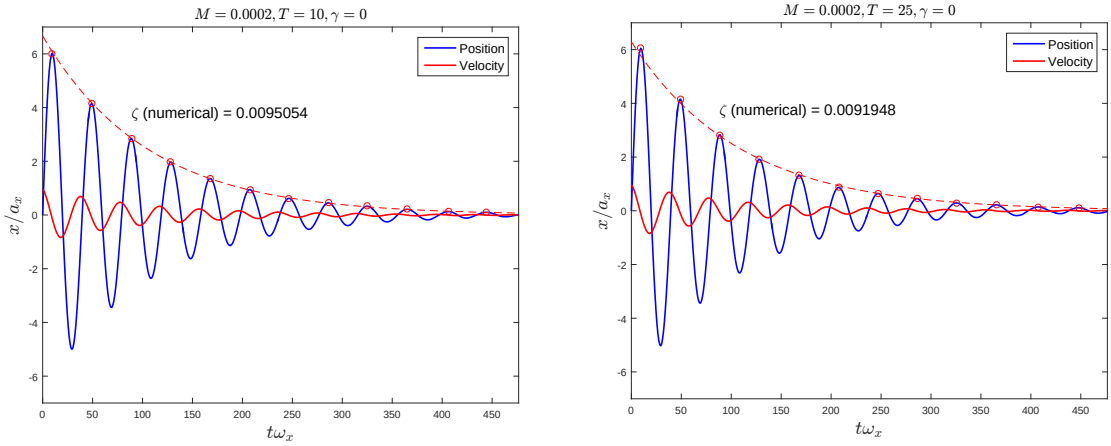


(a) (b)

Figure 5.1: Damped Kohn mode oscillations for: (a) $T = 0$, $\mathcal{M} = 0.0002$, and (b) $T = 0$, $\mathcal{M} = 0.0001$. The analytical and numerical modes are plotted simultaneously showing good agreement.

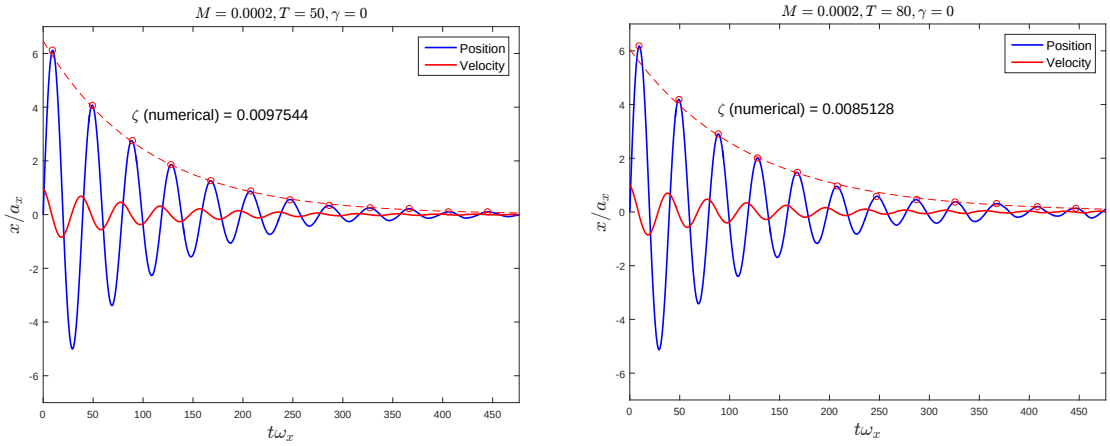
be $\zeta_{\text{analytic}} = 0.010496$ compared with the numerical value of $\zeta_{\text{numeric}} = 0.009583$, that is a difference of about 8.7%. Furthermore for the case of Fig. 5.1(b), $\zeta_{\text{analytic}} = 0.0052479$ and $\zeta_{\text{analytic}} = 0.0047713$, a difference of about 9.1%. So the decay rate which corresponds to how rapidly energy is lost by the condensate as it moves in the thermal cloud is determinate.

Thermal effects and fluctuations become important at finite temperature and the damping rate is seen to be influenced by temperature as can be seen in Fig (5.2).



(a) $T = 10$

(b) $T = 25$



(c) $T = 50$

(d) $T = 80$

Figure 5.2: Kohn mode oscillations for a range of temperatures.

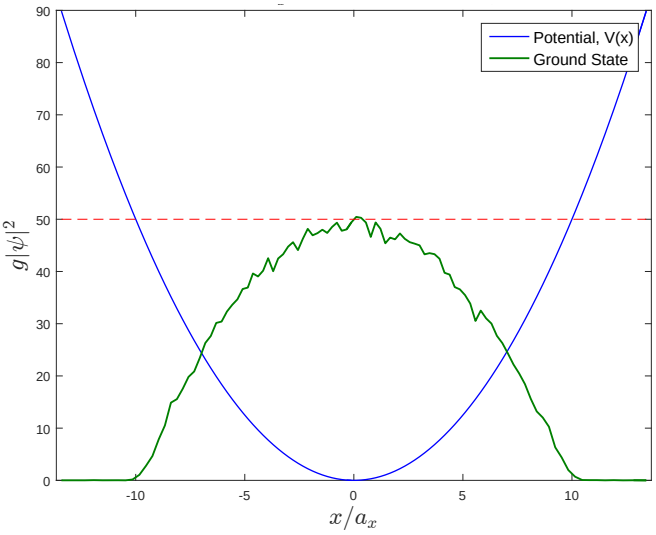


Figure 5.3: High temperature $T = 80$ thermalised state showing noisy thermal fluctuations due evolved using the 1D SPGPE.

Chapter 6

Conclusion

This overall project involved looking into the energy damping reservoir interactions between the coherent region or C-field region, characterising the condensate and the non-C-field region of the thermal cloud. In principle the equations of motion for the C-field, i.e. the condensate was derived but the further details of it will be left to some other instance of time. In particular the coupling between the reservoir and the matter wave density causing dissipation of the condensate motion. However in running the 1D SPGPE using the Thomas-Fermi groundstate, the noisy and thermal fluctuations were hard to resolve and observe, as only very large amplitudes of oscillations and an extended period of run time would show clear distortions of the condensate path. This was not clearly observed, but what was observed was subtle.

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