UNIVERSITY OF OTAGO EXAMINATIONS 2013

PHYSICS

Physics module 401

Linear Systems

Semester One

(TIME ALLOWED: 2 HOURS)

This examination paper comprises 5 pages

Candidates should answer questions as follows:

Answer TWO out of the THREE questions.
Questions carry equal weight.

The following material is provided:

Use of calculators:

No restriction on the model of calculator to be used, but no device with communication capability shall be accepted as a calculator.
(Subject to inspection by the examiners.)

Candidates are permitted copies of:

No additional material.

Other Instructions:

DO NOT USE RED INK OR PENCIL.
USEFUL RELATIONSHIPS can be found on page 2.

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A table of Fourier transforms and properties

Forward: $F(\nu) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{j}2\pi\nu t} dt$ Inverse: $f(t) = \int_{-\infty}^{\infty} F(\nu)\, \mathrm{e}^{\mathrm{j}2\pi\nu t} d\nu$

Some properties

$F(t)$	$\leftrightarrow$	$f(-\nu)$
$f^*(t)$	$\leftrightarrow$	$F^*(-\nu)$
$f(at)$	$\leftrightarrow$	$\frac{1}{\lvert a\rvert} F\left(\frac{\nu}{a}\right)$
$f(t-t_0)$	$\leftrightarrow$	$\mathrm{e}^{-\mathrm{j}2\pi\nu t_0} F(\nu)$
$\mathrm{e}^{\mathrm{j}2\pi\nu_0 t} f(t)$	$\leftrightarrow$	$F(\nu-\nu_0)$
$\frac{d^n}{dt^n} f(t)$	$\leftrightarrow$	$(\mathrm{j}2\pi\nu)^n F(\nu)$
$-\mathrm{j}2\pi t f(t)$	$\leftrightarrow$	$\frac{dF(\nu)}{d\nu}$
$\int_{-\infty}^{t} f(\tau)\, d\tau$	$\leftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu} F(\nu) + \frac{1}{2} F(0)\, \delta(\nu)$
$(f * g)(t)$	$\leftrightarrow$	$F(\nu)\, G(\nu)$
$f(t)\, g(t)$	$\leftrightarrow$	$(F * G)(\nu)$

Some transform pairs

$\delta(t)$	$\leftrightarrow$	1
$u(t)\, \mathrm{e}^{-at}$	$\leftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu + a}$
$u(t)$	$\leftrightarrow$	$\frac{1}{2}\delta(\nu) + \frac{1}{\mathrm{j}2\pi\nu}$
$\exp(\mathrm{j}2\pi\nu_0 t)$	$\leftrightarrow$	$\delta(\nu-\nu_0)$
$\cos(2\pi\nu_0 t)$	$\leftrightarrow$	$\frac{1}{2}\left[\delta(\nu-\nu_0) + \delta(\nu+\nu_0)\right]$
$\sin(2\pi\nu_0 t)$	$\leftrightarrow$	$\frac{\mathrm{j}}{2}\left[-\delta(\nu-\nu_0) + \delta(\nu+\nu_0)\right]$
$\Pi(t)$	$\leftrightarrow$	$\mathrm{sinc}(\nu)$
$\mathrm{sgn}(t)$	$\leftrightarrow$	$\frac{1}{\mathrm{j}\pi\nu}$
$\sum_{k=-\infty}^{\infty} \delta(t-kT)$	$\leftrightarrow$	$\frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(\nu - \frac{k}{T}\right)$
$\exp(-\pi t^2)$	$\leftrightarrow$	$\exp(-\pi\nu^2)$

DFT: $X[r] = \frac{1}{N} \sum_{k=0}^{N-1} x[k] \exp\left(-\frac{\mathrm{j}2\pi rk}{N}\right)$ IDFT: $x[k] = \sum_{r=0}^{N-1} X[r] \exp\left(\frac{\mathrm{j}2\pi rk}{N}\right)$

Hilbert transform: $\hat{f}(t) = f(t) * \frac{1}{\pi t}$

Convolution integral: $(f * h)(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, \mathrm{d}\tau$

1. (a) Use $f_1(t)$ and $f_2(t)$ to denote arbitrary inputs and $g_1(t)$ and $g_2(t)$ to denote the respective outputs. Describe what it means for a system to be:

   (i) Linear

   (ii) Causal

   (iii) Memoryless

   (iv) Time-invariant

   (b) Show that $\exp(j2\pi\nu_0 t)$ is an eigen-function of a linear time invariant system. What is the eigenvalue?

   (c) Calculate the convolution of $\mathrm{sinc}(t)$ and $\sin(20\pi t)\,\mathrm{sinc}(t)$.

   (d) The function $f(t)$ is real valued, even, and has Fourier transform $F(\nu)$. Find the real part of the Fourier transform of

$$g(t) = u(t)f(t).$$

   (Here $u(t)$ is the unit step function)

   (e) A system is said to be BIBO-stable if every bounded input leads to a bounded output.

   (i) Show that a system with an impulse response function that is absolutely integrable is BIBO stable.

   (ii) From the definition of BIBO-stability, and a carefully chosen input function, show that the system with the impulse response function

$$h(t) = \frac{u(t)\sin(t)}{t}$$

   is not BIBO-stable.

   (f) Either by using the properties of convolution or directly from the definition, show that:

   If

$$F(t) = \int_{-\infty}^{t} f(\tau)\,d\tau,$$

   then

$$(F * g)(t) = \int_{-\infty}^{t} (f * g)(\tau)\,d\tau\,.$$

2. (a) Prove Parseval's theorem, which is that

$$\int_{-\infty}^{\infty} f(t)g^*(t)\,dt = \int_{-\infty}^{\infty} F(\nu)g^*(\nu)\,d\nu.$$

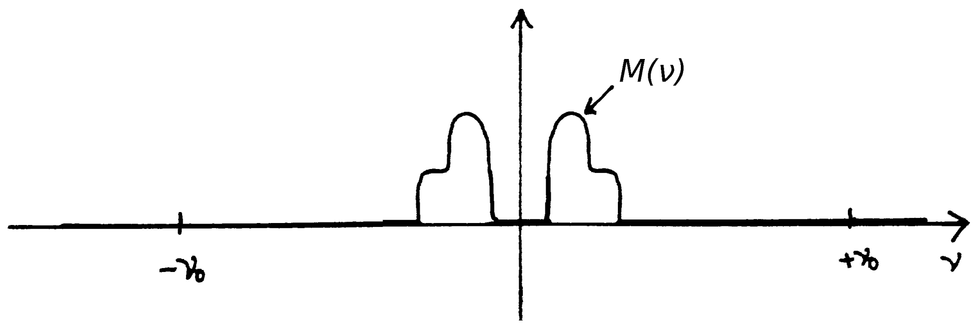
Here the Fourier transforms of $f(t)$ and $g(t)$ are $F(\nu)$ and $G(\nu)$, respectively. You may use any of the results on page 2.

(b) Suppose that $m(t)$ is a real valued function that has analytic signal $m_a(t)$, and that

$$f(t) = \mathrm{Re}[m_a(t)\exp(j2\pi\nu_0 t)].$$

Derive an expression for $F(\nu)$, the Fourier transform of $f(t)$. Write your answer in terms of $M(\nu)$ the Fourier transform of $m(t)$.

Copy the following graph into your answer booklet and add a sketch of $F(\nu)$ to it.



(c) Using the fact that $F_a(\nu) = 2u(\nu)F(\nu)$, or otherwise, show that a function $f(t)$ and its Hilbert transform $\hat{f}(t)$ have the same energy spectral density.

(d) The power auto-correlation function of a function $f(t)$ is defined as

$$\phi^p_{ff}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f^*(t)f(t+\tau)\,d\tau.$$

(i) State the Wiener-Khinchin Theorem.

(ii) Prove the Wiener-Khinchin theorem by considering the average output power after $f(t)$ is passed through an ideal bandpass filter.

TURN OVER

3. (a) (i) State formally using test functions the definition of the Fourier transform for generalised functions.

(ii) Suppose that the generalised function $G(\nu)$ is the Fourier transform of the generalised function $g(t)$. Starting from the above definition, show that

$$g'(t) \leftrightarrow j2\pi\nu G(\nu).$$

You may assume that the properties on page 2 hold for test functions.

(b) Show using their actions on test functions that

(i) $u'(t) = \delta(t)$

(ii) $f(t)\delta'(t) = f'(0)\delta(t) - f(0)\delta'(t)$

(c) A function $s(t)$ is uniformly sampled every T seconds to give the sequence

$$\{s[k] : s[k] = s(kT)\}.$$

(i) State the conditions given in the *Sampling Theorem* that are required for $s(t)$ to be determined from its samples $s[k]$.

(ii) Calculate the Fourier transform of

$$s_s(t) = s(t) \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

(iii) From your answer in (ii), derive a method for calculating $s(t)$ from $s[n]$ in the situation where the conditions of the sampling theorem are satisfied. Illustrate your answer with diagrams where appropriate.

(iv) Suppose that $s(t)$ is periodic and repeats every NT, where N is a positive integer. (This means that for all k, we have that $s[k] = s[k \bmod n]$.) Calculate the Fourier transform of $s_s(t)$ for this periodic case, and express your answer in terms of $S[r]$ the discrete Fourier transform of $s[k]$.

END