UNIVERSITY OF OTAGO EXAMINATIONS 2015

PHYSICS

Physics Module 401

Linear Systems

Semester One

(TIME ALLOWED: 2 HOURS)

This examination paper comprises 5 pages

Candidates should answer questions as follows:

Answer TWO out of the THREE questions.
Questions carry equal weight.

The following material is provided:

Use of calculators:

No restriction on the model of calculator to be used, but no device with communication capability shall be accepted as a calculator.
(Subject to inspection by the examiners.)

Candidates are permitted copies of:

No additional material.

Other Instructions:

DO NOT USE RED INK OR PENCIL.
USEFUL RELATIONSHIPS can be found on page 2.

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A table of Fourier transforms and properties

Forward: $F(\nu) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{j}2\pi\nu t} dt$ Inverse: $f(t) = \int_{-\infty}^{\infty} F(\nu)\, \mathrm{e}^{\mathrm{j}2\pi\nu t} d\nu$

Some properties

$F(t)$	$\leftrightarrow$	$f(-\nu)$
$f^*(t)$	$\leftrightarrow$	$F^*(-\nu)$
$f(at)$	$\leftrightarrow$	$\frac{1}{\lvert a\rvert} F\left(\frac{\nu}{a}\right)$
$f(t-t_0)$	$\leftrightarrow$	$\mathrm{e}^{-\mathrm{j}2\pi\nu t_0} F(\nu)$
$\mathrm{e}^{\mathrm{j}2\pi\nu_0 t} f(t)$	$\leftrightarrow$	$F(\nu-\nu_0)$
$\frac{d^n}{dt^n} f(t)$	$\leftrightarrow$	$(\mathrm{j}2\pi\nu)^n F(\nu)$
$-\mathrm{j}2\pi t f(t)$	$\leftrightarrow$	$\frac{dF(\nu)}{d\nu}$
$\int_{-\infty}^{t} f(\tau)\, d\tau$	$\leftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu} F(\nu) + \frac{1}{2} F(0)\, \delta(\nu)$
$(f * g)(t)$	$\leftrightarrow$	$F(\nu)\, G(\nu)$
$f(t)\, g(t)$	$\leftrightarrow$	$(F * G)(\nu)$

Some transform pairs

$\delta(t)$	$\leftrightarrow$	1
$u(t)\, \mathrm{e}^{-at}$	$\leftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu + a}$
$u(t)$	$\leftrightarrow$	$\frac{1}{2}\delta(\nu) + \frac{1}{\mathrm{j}2\pi\nu}$
$\exp(\mathrm{j}2\pi\nu_0 t)$	$\leftrightarrow$	$\delta(\nu-\nu_0)$
$\cos(2\pi\nu_0 t)$	$\leftrightarrow$	$\frac{1}{2}\left[\delta(\nu-\nu_0) + \delta(\nu+\nu_0)\right]$
$\sin(2\pi\nu_0 t)$	$\leftrightarrow$	$\frac{\mathrm{j}}{2}\left[-\delta(\nu-\nu_0) + \delta(\nu+\nu_0)\right]$
$\Pi(t)$	$\leftrightarrow$	$\mathrm{sinc}(\nu)$
$\mathrm{sgn}(t)$	$\leftrightarrow$	$\frac{1}{\mathrm{j}\pi\nu}$
$\sum_{k=-\infty}^{\infty} \delta(t-kT)$	$\leftrightarrow$	$\frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\left(\nu - \frac{k}{T}\right)$
$\exp(-\pi t^2)$	$\leftrightarrow$	$\exp(-\pi\nu^2)$

DFT: $X[r] = \dfrac{1}{N}\sum_{k=0}^{N-1} x[k] \exp\left(-\dfrac{\mathrm{j}2\pi rk}{N}\right)$ IDFT: $x[k] = \sum_{r=0}^{N-1} X[r] \exp\left(\dfrac{\mathrm{j}2\pi rk}{N}\right)$

Hilbert transform: $\hat{f}(t) = f(t) * \dfrac{1}{\pi t}$

Convolution integral: $(f * h)(t) = \displaystyle\int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, \mathrm{d}\tau$

1. (a) Give definitions for the following:
   (i) A linear system.
   (ii) A time invariant system.
   (iii) A causal system.
   (iv) The impulse response for a linear time invariant system.

   (b) Show that $\exp(j2\pi\nu_o t)$ is an eigenfunction of a linear time invariant system and derive an expression for the eigenvalue in terms of the impulse response.

   (c) Give the Fourier transforms of the following functions
   (i) $g(t) = \mathrm{sinc}(10t)\cos(2\pi t)$,
   (ii) $h(t) = \exp(-5|t|)$,
   (iii) $b(t) = |\cos(\pi t)|$.

   Hint: $\sum_{k=-\infty}^{\infty} \delta\left(t - kT\right) \leftrightarrow \frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\left(\nu - \frac{k}{T}\right)$

   (d) Show that:
   (i) $f * g = g * f$,
   (ii) $(f * g)' = f' * g = f * g'$.

   (e) A complex valued function $f(t)$ has Fourier transform $F(\nu)$, and can be written as

$$f(t) = x(t) + jy(t)$$

   where both of the $x(t)$ and $y(t)$ are real. Determine an expression for each of $X(\nu)$ and $Y(\nu)$ in terms of $F(\nu)$

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2. (a) Starting from Parseval's theorem prove the convolution theorem.

(b) For a real-valued function $f(t)$ the analytic signal is given by

$$F_a(\nu) = 2u(\nu)F(\nu)$$

and the Hilbert transform, $\hat{f}(t)$, by $f_a(t) = f(t) + j\hat{f}(t)$.
Show that

$$\hat{f}(t) = \int_{-\infty}^{\infty} \frac{f(\tau)}{\pi(t-\tau)} d\tau.$$

(c) Given a linear time invariant system with impulse response response function

$$h(t) = u(t)\exp(-t).$$

(i) Give the definition for what it means for a linear time invariant system to be stable.

(ii) State the stability theorem.

(iii) Is the system stable?

(iv) Prove the stability or otherwise of this system starting from the definition of stability.

(d) Consider a function $w(t)$ that is periodic with period T, in other words for all integers k and time values t we have $w(t) = w(t + kT)$.
Show that:

$$w(t) = \sum_{k=-\infty}^{\infty} c_k \exp(j2\pi kt/T)$$

and give an expression for the c_k.

3. (a) The generalized function $f(t)$ has Fourier transform $F(\nu)$.
   (i) Give the formal definition describing what this means.
   (ii) Show from this definition that the Fourier transform of $\frac{d}{dt}f(t)$ is $j2\pi\nu F(\nu)$. For the purposes of this question, you can assume that any *test* function and its Fourier transform follow the properties given on page 2.

   (b) A stationary stochastic process $f(t)$ passes through a linear time invariant system with impulse response $h(t)$, resulting in output $g(t)$.
   (i) Show that the cross correlation function satisfies

$$\phi_{fg}(\tau) = (\phi_{ff} * h)(\tau)$$

   where $\phi_{fg}(\tau) = E[f^*(t)g(t+\tau)]$.
   (ii) In a similar fashion it can be shown (you don't have to) that

$$\phi_{gg}(\tau) = (\phi_{ff} * h * \tilde{h})(\tau)$$

   where $\tilde{h}(t) = h^*(-t)$.
   Based on this result provide an argument why $\Phi_{ff}(\nu)$, the Fourier transform of $\phi_{ff}(\tau)$, should be considered the power spectral density of f.

   (c) White noise $n(t)$ with spectrum $\Phi_{nn}(\nu) = N/2$ is input to an ideal low pass filter that has transfer function $H(\nu) = \Pi(\nu T)$. The resulting output signal is sampled at time values $... - 3T, -2T, -T, 0, T, 2T, 3T....$ Show that the resulting samples are are independent random numbers. What is their variance?

END