UNIVERSITY OF OTAGO EXAMINATIONS 2014

PHYSICS

Physics Module 401

Linear Systems

Semester One

(TIME ALLOWED: 2 HOURS)

This examination paper comprises 5 pages

Candidates should answer questions as follows:

Answer TWO out of the THREE questions.
Questions carry equal weight.

The following material is provided:

Use of calculators:

No restriction on the model of calculator to be used, but no device with communication capability shall be accepted as a calculator.
(Subject to inspection by the examiners.)

Candidates are permitted copies of:

No additional material.

Other Instructions:

DO NOT USE RED INK OR PENCIL.
USEFUL RELATIONSHIPS can be found on page 2.

TURN OVER

A table of Fourier transforms and properties

Forward: $F(\nu) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{j}2\pi\nu t} dt$ Inverse: $f(t) = \int_{-\infty}^{\infty} F(\nu)\, \mathrm{e}^{\mathrm{j}2\pi\nu t} d\nu$

Some properties			Some transform pairs		
$F(t)$	$\leftrightarrow$	$f(-\nu)$	$\delta(t)$	$\leftrightarrow$	1
$f^*(t)$	$\leftrightarrow$	$F^*(-\nu)$	$u(t)\,\mathrm{e}^{-at}$	$\leftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu + a}$
$f(at)$	$\leftrightarrow$	$\frac{1}{\lvert a\rvert} F\left(\frac{\nu}{a}\right)$	$u(t)$	$\leftrightarrow$	$\frac{1}{2}\delta(\nu) + \frac{1}{\mathrm{j}2\pi\nu}$
$f(t - t_0)$	$\leftrightarrow$	$\mathrm{e}^{-\mathrm{j}2\pi\nu t_0} F(\nu)$	$\exp(\mathrm{j}2\pi\nu_0 t)$	$\leftrightarrow$	$\delta(\nu - \nu_0)$
$\mathrm{e}^{\mathrm{j}2\pi\nu_0 t} f(t)$	$\leftrightarrow$	$F(\nu - \nu_0)$	$\cos(2\pi\nu_0 t)$	$\leftrightarrow$	$\frac{1}{2}\left[\delta(\nu - \nu_0) + \delta(\nu + \nu_0)\right]$
$\frac{d^n}{dt^n} f(t)$	$\leftrightarrow$	$(\mathrm{j}2\pi\nu)^n F(\nu)$	$\sin(2\pi\nu_0 t)$	$\leftrightarrow$	$\frac{\mathrm{j}}{2}\left[-\delta(\nu - \nu_0) + \delta(\nu + \nu_0)\right]$
$-\mathrm{j}2\pi t f(t)$	$\leftrightarrow$	$\frac{dF(\nu)}{d\nu}$	$\Pi(t)$	$\leftrightarrow$	$\mathrm{sinc}(\nu)$
$\int_{-\infty}^{t} f(\tau)\, d\tau$	$\leftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu} F(\nu) + \frac{1}{2} F(0)\,\delta(\nu)$	$\mathrm{sgn}(t)$	$\leftrightarrow$	$\frac{1}{\mathrm{j}\pi\nu}$
$(f * g)(t)$	$\leftrightarrow$	$F(\nu)\, G(\nu)$	$\sum_{k=-\infty}^{\infty} \delta(t - kT)$	$\leftrightarrow$	$\frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(\nu - \frac{k}{T}\right)$
$f(t)\, g(t)$	$\leftrightarrow$	$(F * G)(\nu)$	$\exp(-\pi t^2)$	$\leftrightarrow$	$\exp(-\pi\nu^2)$

DFT: $X[r] = \frac{1}{N} \sum_{k=0}^{N-1} x[k] \exp\left(-\frac{\mathrm{j}2\pi rk}{N}\right)$ IDFT: $x[k] = \sum_{r=0}^{N-1} X[r] \exp\left(\frac{\mathrm{j}2\pi rk}{N}\right)$

Hilbert transform: $\hat{f}(t) = f(t) * \frac{1}{\pi t}$

Convolution integral: $(f * h)(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t - \tau)\, \mathrm{d}\tau$

TURN OVER

1. (a) Give definitions for:
   - (i) What it means for a system to be linear.
   - (ii) What it means for a system to be shift invariant.
   - (iii) The impulse response of a linear system.
   - (iv) The transfer function for a linear shift invariant system.

   (b) Why is the transfer function not a useful concept for a linear system that is not shift invariant?

   (c) Derive expressions for the Fourier transforms of the following functions. Do so directly from the definition of the Fourier transform and use $F(\nu)$ to denote the Fourier transform of $f(t)$.
   - (i) $tf(t)$
   - (ii) $f'(t)$
   - (iii) $f(\alpha t)$ where $\alpha > 0$

   (d) Give an approximation to the convolution of $\exp(-\pi t^2)$ and $\sin(40\pi t)\exp(-\pi t^2)$. Explain your answer fully.

   (e) Prove the following equalities:

$$\frac{d}{dt}(f * g)(t) = (f' * g) = (f * g')$$

2. (a) Using their actions on test functions show that the derivative of the unit step is the delta function.

   (b) A system h, which produces diminishing echoes, has the impulse response

$$h(t) = \sum_{k=0}^{\infty} a^k \delta(t - kT)$$

   where $0 < a < 1$.

   (i) Is this system stable? Fully explain your answer.

   (ii) Prove that the transfer function satisfies

$$H(\nu) = H\left(\nu + \frac{n}{T}\right)$$

   for all integers n.

   (c) A white noise signal with a power spectral density given by $\Phi_{xx}(\nu) = N_0/2$ is passed through a filter with transfer function $H(\nu) = \Pi(\nu T)$

   (i) What is the power spectral density of the output?

   (ii) What is the auto correlation function for the output?

   (iii) Show that if the output is sampled every T the result is random numbers that are independent of one another. What is their variance?

   (d) What does it mean for a random process to be ergodic? Give a real world example of an ensemble of functions of time, that is:

   (i) ergodic

   (ii) not ergodic.

3. (a) State and prove Rayleigh's theorem for energy invariance. You may use any of the results on page 2.

(b) Using Rayleigh's theorem show that a real valued function $f(t)$, its analytic signal $f_a(t)$ and its Hilbert transform $\hat{f}(t)$ satisfy

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |\hat{f}(t)|^2\, dt = \frac{1}{2}\int_{-\infty}^{\infty} |f_a(t)|^2\, dt$$

(c) The causal system h, has an impulse response function, $h(t)$, which has no singularities at $t = 0$.

(i) Show that

$$H(\nu) = -\frac{j}{\pi}\int_{-\infty}^{\infty} \frac{H(s)}{\nu - s}\, ds$$

(ii) And hence derive the Kramers-Kronig relations between the real and imaginary parts of $H(\nu)$.

(d) For an arbitrary function $f(t)$ we can define the Hermitian and anti-Hermitian parts via

$$f_h(t) = \frac{1}{2}\big(f(t) + f^*(-t)\big), \quad \text{and} \quad f_{ah}(t) = \frac{1}{2}\big(f(t) - f^*(-t)\big).$$

Show that

$$\mathcal{F}\big(f_h(t)\big) = \mathrm{Re}\big[F(\nu)\big], \quad \text{and} \quad \mathcal{F}\big(f_{ah}(t)\big) = j\,\mathrm{Im}\big[F(\nu)\big].$$

(e) Now consider an $f(t)$ that is real, is zero for $t < 0$, and has with no singularities at $t = 0$.

(i) Show that $f(t) = 2u(t)f_h(t)$.

(ii) Suppose that we also have $\mathrm{Re}\big[F(\nu)\big] = \mathrm{sinc}(\nu)$, find $f(t)$ and $\mathrm{Im}\big[F(\nu)\big]$.

END