UNIVERSITY OF OTAGO EXAMINATIONS 2020

PHYSICS

ELEC441

Linear Systems and Noise
Semester One

(TIME ALLOWED: 3 HOURS)

Online exam platform: Exam distributed via the course webpage.

Candidates should answer questions as follows:

Answer TWO out of the THREE questions.
Questions carry equal weight.

The following material is provided:

A table of useful formulae is given on page 4.

Use of calculators:

No restriction on the model of calculator to be used, but no device with communication capability shall be accepted as a calculator.

Candidates are permitted copies of:

All course material and any reference material and notes, but these must be available locally on your device, as a book or paper notes.

Other Instructions:

DO NOT USE RED INK OR PENCIL.
Symbols for physical quantities are given in italics.
Symbols for vector quantities are in bold.
You must read the Academic Integrity Notice on page 2 before starting this exam.

UNIVERSITY of OTAGO
Te Whare Wānanga o Otāgo
NEW ZEALAND

Online Assessment

Academic Integrity Notice

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- Communicating with any other person (except to report technical difficulties to a University staff member or representative)
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- Copying pre-prepared text as answers
- Exceeding the specified time limit for the assessment

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By completing this assessment, you are taken to have read and understood this notice.

TURN OVER

How To Get Help

IMPORTANT - PLEASE READ

Technical Help

In the event of experiencing IT or other technical problems during any part of this examination, immediately contact AskOtago via online chat at
https://otago.custhelp.com/
or phone toll-free in NZ: 0800 80 80 98
or email ask@otago.ac.nz

A table of useful formulae

Forward: $F(\nu) = \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-\mathrm{j}2\pi\nu t}dt$ Inverse: $f(t) = \int_{-\infty}^{\infty} F(\nu)\,\mathrm{e}^{\mathrm{j}2\pi\nu t}d\nu$

Some properties		
$F(t)$	$\longleftrightarrow$	$f(-\nu)$
$f^*(t)$	$\longleftrightarrow$	$F^*(-\nu)$
$f(at)$	$\longleftrightarrow$	$\frac{1}{\lvert a\rvert}F\left(\frac{\nu}{a}\right)$
$f(t-t_0)$	$\longleftrightarrow$	$\mathrm{e}^{-\mathrm{j}2\pi\nu t_0}F(\nu)$
$\mathrm{e}^{\mathrm{j}2\pi\nu_0 t}f(t)$	$\longleftrightarrow$	$F(\nu-\nu_0)$
$\frac{d^n}{dt^n}f(t)$	$\longleftrightarrow$	$(\mathrm{j}2\pi\nu)^n F(\nu)$
$(-\mathrm{j}2\pi t)^n f(t)$	$\longleftrightarrow$	$\frac{d^nF(\nu)}{d\nu^n}$
$\int_{-\infty}^{t} f(\tau)\,d\tau$	$\longleftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu}F(\nu) + \frac{1}{2}F(0)\,\delta(\nu)$
$(f*g)(t)$	$\longleftrightarrow$	$F(\nu)\,G(\nu)$
$f(t)\,g(t)$	$\longleftrightarrow$	$(F*G)(\nu)$

Some transform pairs		
$\delta(t)$	$\longleftrightarrow$	1
$u(t)\,\mathrm{e}^{-at}$	$\longleftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu+a}$
$u(t)$	$\longleftrightarrow$	$\frac{1}{2}\delta(\nu) + \frac{1}{\mathrm{j}2\pi\nu}$
$\exp(\mathrm{j}2\pi\nu_0 t)$	$\longleftrightarrow$	$\delta(\nu-\nu_0)$
$\cos(2\pi\nu_0 t)$	$\longleftrightarrow$	$\frac{1}{2}\left[\delta(\nu-\nu_0)+\delta(\nu+\nu_0)\right]$
$\sin(2\pi\nu_0 t)$	$\longleftrightarrow$	$\frac{1}{2\mathrm{j}}\left[\delta(\nu-\nu_0)-\delta(\nu+\nu_0)\right]$
$\Pi(t)$	$\longleftrightarrow$	$\mathrm{sinc}(\nu)$
$\mathrm{sgn}(t)$	$\longleftrightarrow$	$\frac{1}{\mathrm{j}\pi\nu}$
$\exp(-\pi t^2)$	$\longleftrightarrow$	$\exp(-\pi\nu^2)$
$\Lambda(t)$	$\longleftrightarrow$	$\mathrm{sinc}(\nu)^2$
$e^{-a\lvert t\rvert}$	$\longleftrightarrow$	$\frac{2a}{a^2+(2\pi\nu)^2}$
$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\longleftrightarrow$	$\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(\nu-\frac{k}{T}\right)$

DFT: $X[r] = \sum_{k=0}^{N-1} x[k]\exp\left(-\frac{\mathrm{j}2\pi rk}{N}\right)$ IDFT: $x[k] = \frac{1}{N}\sum_{r=0}^{N-1} X[r]\exp\left(\frac{\mathrm{j}2\pi rk}{N}\right)$

Hilbert transform: $\hat{f}(t) = f(t) * \frac{1}{\pi t}$

Convolution integral: $(f*h)(t) = \int_{-\infty}^{\infty} f(\tau)\,h(t-\tau)\;\mathrm{d}\tau$

Cauchy-Schwarz inequality: $\left|\int_{-\infty}^{\infty} f(\tau)g(\tau)d\tau\right|^2 \le \int_{-\infty}^{\infty}|f(\tau)|^2d\tau\ \int_{-\infty}^{\infty}|g(\tau)|^2d\tau$

Geometric series: $\sum_{k=0}^{N-1} z^k = \frac{1-z^N}{1-z},\quad z \ne 1;\qquad \sum_{k=0}^{\infty} z^k = \frac{1}{1-z},\quad |z|<1.$

$$\lim_{N\to\infty}\left(1+\frac{x}{N}\right)^N = \exp(x)$$

1. (a) Consider a system with input $f(t)$ and output $g(t)$.

   (i) What does it mean for the system to be

   i. linear,

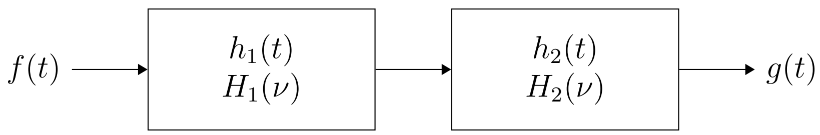
   ii. time-shift invariant.

   Describe any simplifications that may be applied to the impulse response of a linear, time-shift invariant (LTI) system. (3 marks)

   (ii) Using linearity and the decomposition property of the delta function, show that the output $g(t)$ of an LTI system driven by input $f(t)$ can be written as $g(t) = (f * h)(t)$, where $h(t)$ is the impulse response of the system. (3 marks)

   (iii) Consider an LTI system with transfer function $H(\nu)$. Show that $f(t) = \exp(j2\pi\nu t)$ is an eigenfunction of the system, and derive an expression for the eigenvalue in terms of $H(\nu)$. (2 marks)

(b) Consider a cascaded system constructed by feeding the output of LTI subsystem 1 into LTI subsystem 2:



   (i) Show that the transfer function for the cascaded system is $H(\nu) = H_1(\nu)H_2(\nu)$. (2 marks)

   (ii) Calculate the output $g(t)$ of the cascaded system comprised of $h_1(t) = u(t)e^{-at}$, and

$$h_2(t) = e^{-b|t|}\left[\Lambda(t) * \sum_{k=-\infty}^{\infty} \delta(t-k)\right]$$

   when the system is driven by the input $f(t) = \exp(\mathrm{j}2\pi\nu_0 t)$. What is the phase change applied to the input? (4 marks)

(c) A chain of N subsystems, each with impulse response $h(t) = \delta(t) - \delta'(t)\Delta t$, forms a cascaded system, where Δt is a real number.

   (i) Derive the transfer function for $h(t)$. (3 marks)

   (ii) Setting $T = N\Delta t$ constant, and taking the limit $N \to \infty$, derive an expression for the total transfer function of the cascaded system. What is the output for input $f(t) = \exp(\mathrm{j}2\pi\nu_0 t)$? (3 marks)

2. (a) (i) Give a definition of (BIBO) stability for a system in terms of its inputs and outputs. (2 marks)

(ii) Show that a linear, time-shift invariant (LTI) system is stable provided its impulse response $h(t)$ is absolutely integrable. (3 marks)

(b) (i) Consider an LTI system with impulse response

$$h(t) = \sum_{k=0}^{N-1} \delta(t - kT) z^k,$$

for complex z, with $|z| \neq 1$. This generates a finite series of scaled, phase-shifted echoes. Show that the system is stable for any finite $|z|$. (3 marks)

(ii) Derive an expression for the transfer function of the system. What is the output for $f(t) = \exp(\mathrm{j}2\pi\nu_0 t)$? Write your answers in terms of z. (2 marks)

(c) (i) Consider an LTI system with impulse response

$$h(t) = \sum_{k=0}^{\infty} \delta(t - kT) z^k,$$

for $z = r\exp(\mathrm{j}2\pi\nu_0 T)$, and real parameters $0 < r < \infty$, ν_0. This generates an *infinite* series of scaled, phase-shifted echoes. Show that the system is stable provided $r < 1$. (3 marks)

(ii) Show that the transfer function for the system is

$$H(\nu) = \frac{1}{1 - re^{\mathrm{j}2\pi(\nu_0 - \nu)T}}$$

(3 marks)

(iii) For the resonant input $f(t) = \exp(\mathrm{j}2\pi\nu_0 t)$, find an expression for the output $g(t)$ in terms of $f(t)$ and the transfer function. Explain what happens as $r \to 1$.

(2 marks)

(iv) Consider an anti-resonant input $f(t) = \exp(\mathrm{j}2\pi(\nu_0 - 1/(2T))t)$. Find an expression for the output and explain the result with reference to (e), paying attention to the $r \to 1$ limit.

(2 marks)

3. (a) Consider a generalized function $g(t)$, and open support test function $\phi(t)$, with $\phi(t) \longleftrightarrow \Phi(\nu)$.

   (i) Give the definition of the Fourier transform $G(\nu)$ of $g(t)$, in terms of its action on open support test functions. Define any notation you introduce.
   (3 marks)

   (ii) Using the definition, derive the Fourier transform of

   i. $f(t) = \sin(2\pi\nu_0 t)$,

   ii. $g(t) = \delta'(t)$,

   iii. $h(t) = \delta'(t)f(t)$, for any slowly increasing $f(t)$.

   You may assume test functions satisfy any Fourier transform property listed on page 4. (7 marks)

(b) Derive the Fourier transform of $f(t) = |\sin(t)|$. (5 marks)

(c) A stationary white noise process with power spectral density

$$\Phi_{nn}(\nu) = N_0,$$

and zero mean is passed through a filter with transfer function $H(\nu) = e^{-b|\nu|}$.

   (i) What is the power-spectral density and average power of the filter output? (2 marks)

   (ii) The output from the filter is now sent through an ideal low pass filter with $H(\nu) = \Pi(\nu T)$. Derive the variance of the filter output. Explain the limits $T \gg b$ and $T \ll b$. (3 marks)

END