UNIVERSITY OF OTAGO EXAMINATIONS 2019

PHYSICS

ELEC441

Linear Systems and Noise
Semester One

(TIME ALLOWED: 2 HOURS)

This examination paper comprises 5 pages

Candidates should answer questions as follows:

Answer TWO out of the THREE questions.
Questions carry equal weight.

The following material is provided:

Use of calculators:

No restriction on the model of calculator to be used, but no device with communication capability shall be accepted as a calculator.
(Subject to inspection by the examiners.)

Candidates are permitted copies of:

No additional material.

Other Instructions:

DO NOT USE RED INK OR PENCIL.
USEFUL RELATIONSHIPS can be found on page 2.

A table of Fourier transforms and properties

Forward: $F(\nu) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{j}2\pi\nu t} dt$ Inverse: $f(t) = \int_{-\infty}^{\infty} F(\nu)\, \mathrm{e}^{\mathrm{j}2\pi\nu t} d\nu$

Some properties

$F(t)$	$\leftrightarrow$	$f(-\nu)$
$f^*(t)$	$\leftrightarrow$	$F^*(-\nu)$
$f(at)$	$\leftrightarrow$	$\frac{1}{\vert a\vert} F\left(\frac{\nu}{a}\right)$
$f(t-t_0)$	$\leftrightarrow$	$\mathrm{e}^{-\mathrm{j}2\pi\nu t_0} F(\nu)$
$\mathrm{e}^{\mathrm{j}2\pi\nu_0 t} f(t)$	$\leftrightarrow$	$F(\nu-\nu_0)$
$\frac{d^n}{dt^n} f(t)$	$\leftrightarrow$	$(\mathrm{j}2\pi\nu)^n F(\nu)$
$-\mathrm{j}2\pi t f(t)$	$\leftrightarrow$	$\frac{dF(\nu)}{d\nu}$
$\int_{-\infty}^{t} f(\tau)\, d\tau$	$\leftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu} F(\nu) + \frac{1}{2} F(0)\, \delta(\nu)$
$(f * g)(t)$	$\leftrightarrow$	$F(\nu)\, G(\nu)$
$f(t)\, g(t)$	$\leftrightarrow$	$(F * G)(\nu)$

Some transform pairs

$\delta(t)$	$\leftrightarrow$	1
$u(t)\, \mathrm{e}^{-at}$	$\leftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu + a}$
$u(t)$	$\leftrightarrow$	$\frac{1}{2}\delta(\nu) + \frac{1}{\mathrm{j}2\pi\nu}$
$\exp(\mathrm{j}2\pi\nu_0 t)$	$\leftrightarrow$	$\delta(\nu-\nu_0)$
$\cos(2\pi\nu_0 t)$	$\leftrightarrow$	$\frac{1}{2}\left[\delta(\nu-\nu_0) + \delta(\nu+\nu_0)\right]$
$\sin(2\pi\nu_0 t)$	$\leftrightarrow$	$\frac{\mathrm{j}}{2}\left[-\delta(\nu-\nu_0) + \delta(\nu+\nu_0)\right]$
$\Pi(t)$	$\leftrightarrow$	$\mathrm{sinc}(\nu)$
$\mathrm{sgn}(t)$	$\leftrightarrow$	$\frac{1}{\mathrm{j}\pi\nu}$
$\sum_{k=-\infty}^{\infty} \delta(t-kT)$	$\leftrightarrow$	$\frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\left(\nu - \frac{k}{T}\right)$
$\exp(-\pi t^2)$	$\leftrightarrow$	$\exp(-\pi\nu^2)$

DFT: $X[r] = \dfrac{1}{N}\displaystyle\sum_{k=0}^{N-1} x[k] \exp\left(-\frac{\mathrm{j}2\pi rk}{N}\right)$ IDFT: $x[k] = \displaystyle\sum_{r=0}^{N-1} X[r] \exp\left(\frac{\mathrm{j}2\pi rk}{N}\right)$

Hilbert transform: $\hat{f}(t) = f(t) * \dfrac{1}{\pi t}$

Convolution integral: $(f * h)(t) = \displaystyle\int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\; \mathrm{d}\tau$

1. (a) What is required of a system for it to be *linear*? (2 marks)

(b) Evaluate

$$\frac{d^2}{dt^2}\left(u(t)\exp(-\alpha t)\right).$$

(Show your working but please note that you are not expected to start from first principles) (3 marks)

(c) White noise with power spectral density $\Phi^p_{nn}(\nu) = \frac{N_0}{2}$ is passed through a filter with impulse response function

$$h(t) = u(t)\exp(-\alpha t).$$

(i) What is the power spectral density of the resulting signal?

(ii) What is the power autocorrelation function of the resulting signal?

(iii) What is the average power in the resulting signal?

(iv) A large number of samples are taken of the resulting signal over a long time interval. What is your best guess for the variance of this set of samples?

The following result might be helpful,

$$\exp(-b|t|) \leftrightarrow \frac{2b}{b^2 + 4\pi^2\nu^2}.$$

(10 marks)

(d) State and prove the convolution theorem.

(5 marks)

2. (a) What is a *causal* sytem? Give the condition that must be satisfied for the impulse response function $h(t|\tau)$ for an *linear causal* system. (2 marks)

(b) For the function

$$f(t) = \cos^2(2\pi\nu_0 t)$$

(i) Sketch the function.

(ii) Calculate the Fourier transform.

(iii) Sketch the Fourier transform. (3 marks)

(c) Show for an open support test function $\phi(t)$, which has Fourier transform $\Phi(\nu)$, that:

(i) $\phi(t - t_0) \leftrightarrow \Phi(\nu)\exp(-j2\pi\nu t_0)$

(ii) $\phi'(t) \leftrightarrow j2\pi\nu F(\nu)$ (4 marks)

(d) Give the definition of the Fourier transform ($G(\nu)$) of a generalised function ($g(t)$). (2 marks)

(e) Starting with this defition show the following results are true for generalised functions. (You may assume any of the results on page 2 of this exam are true for test functions.)

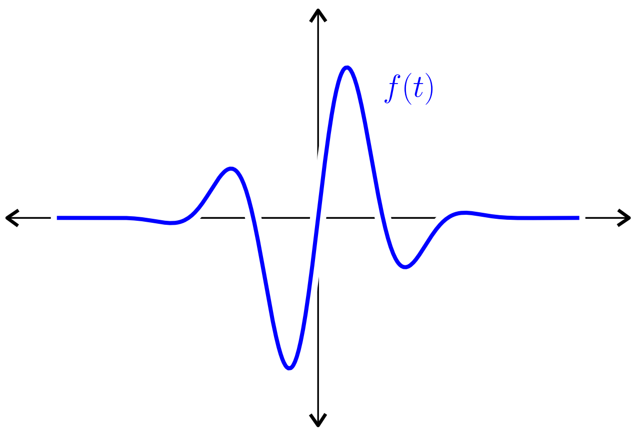
(i) $g(at) \leftrightarrow \frac{1}{|a|}G\left(\frac{\nu}{a}\right)$

(ii) $\delta(t) \leftrightarrow 1$ (6 marks)

(f) Copy the following graph into your answer book and then add in curves for:

(i) The Hilbert transform, $\hat{f}(t)$.

(ii) The absolute value of the analytic signal, $|f_a(t)|$.



(3 marks)

TURN OVER

3. (a) Either using the properties of the convolution or directly from the definition for the convolution show that if

$$F(t) = \int_{-\infty}^{t} f(\tau)\, d\tau,$$

then

$$(F * g)(t) = \int_{-\infty}^{t} (f * g)(\tau)\, d\tau.$$

(4 marks)

(b) A signal $f(t)$ has Fourier transform $F(\nu)$. $f(t)$ is sampled to give

$$f_s(t) = \sum_{k=-\infty}^{\infty} f(kT)\, \delta(t - kT)$$

and then $f_a(t)$, an approximation to $f(t)$ is made by convolving $f_s(t)$ with $\Pi(t/T)$.

(i) Derive an expression for $F_s(\nu)$ in terms of $F(\nu)$

(ii) For a $f(t)$ of your choosing, sketch $f(t)$, $f_s(t)$ and $f_a(t)$

(iii) Derive an expression for $F_a(\nu)$ in terms of $F(\nu)$.

(iv) For a $F(\nu)$ of your choosing, sketch $F(\nu)$, $F_S(\nu)$ and $F_a(\nu)$. (10 marks)

(c) Let $s(t)$ be a complex valued function that has Fourier transform $S(\nu)$. Derive a expression, in terms of $S(\nu)$, for the Fourier transform of $\text{Re}[s(t)]$, the real part of $s(t)$. (2 marks)

(d) Give the definition for (BIBO) stability and show that a linear time invariant system with an absolutely integrable inpulse response function is BIBO stable.

(4 marks)

END

ELEC441

PLEASE DO NOT TURN OVER

THE EXAMINATION PAPER

UNTIL INSTRUCTED TO DO SO