UNIVERSITY OF OTAGO EXAMINATIONS 2018

PHYSICS

ELEC441

Linear Systems and Noise
Semester One

(TIME ALLOWED: 2 HOURS)

This examination paper comprises 5 pages

Candidates should answer questions as follows:

Answer TWO out of the THREE questions.
Questions carry equal weight.

The following material is provided:

Use of calculators:

No restriction on the model of calculator to be used, but no device with communication capability shall be accepted as a calculator.
(Subject to inspection by the examiners.)

Candidates are permitted copies of:

No additional material.

Other Instructions:

DO NOT USE RED INK OR PENCIL.
USEFUL RELATIONSHIPS can be found on page 2.

TURN OVER

A table of Fourier transforms and properties

Forward: $F(\nu) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{j}2\pi\nu t} dt$ Inverse: $f(t) = \int_{-\infty}^{\infty} F(\nu)\, \mathrm{e}^{\mathrm{j}2\pi\nu t} d\nu$

Some properties			Some transform pairs		
$F(t)$	$\leftrightarrow$	$f(-\nu)$	$\delta(t)$	$\leftrightarrow$	1
$f^*(t)$	$\leftrightarrow$	$F^*(-\nu)$	$u(t)\,\mathrm{e}^{-at}$	$\leftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu + a}$
$f(at)$	$\leftrightarrow$	$\frac{1}{\lvert a \rvert} F\left(\frac{\nu}{a}\right)$	$u(t)$	$\leftrightarrow$	$\frac{1}{2}\delta(\nu) + \frac{1}{\mathrm{j}2\pi\nu}$
$f(t - t_0)$	$\leftrightarrow$	$\mathrm{e}^{-\mathrm{j}2\pi\nu t_0} F(\nu)$	$\exp(\mathrm{j}2\pi\nu_0 t)$	$\leftrightarrow$	$\delta(\nu - \nu_0)$
$\mathrm{e}^{\mathrm{j}2\pi\nu_0 t} f(t)$	$\leftrightarrow$	$F(\nu - \nu_0)$	$\cos(2\pi\nu_0 t)$	$\leftrightarrow$	$\frac{1}{2}[\delta(\nu - \nu_0) + \delta(\nu + \nu_0)]$
$\frac{d^n}{dt^n} f(t)$	$\leftrightarrow$	$(\mathrm{j}2\pi\nu)^n F(\nu)$	$\sin(2\pi\nu_0 t)$	$\leftrightarrow$	$\frac{\mathrm{j}}{2}[-\delta(\nu - \nu_0) + \delta(\nu + \nu_0)]$
$-\mathrm{j}2\pi t f(t)$	$\leftrightarrow$	$\frac{dF(\nu)}{d\nu}$	$\Pi(t)$	$\leftrightarrow$	$\mathrm{sinc}(\nu)$
$\int_{-\infty}^{t} f(\tau)\, d\tau$	$\leftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu} F(\nu) + \frac{1}{2} F(0)\,\delta(\nu)$	$\mathrm{sgn}(t)$	$\leftrightarrow$	$\frac{1}{\mathrm{j}\pi\nu}$
$(f * g)(t)$	$\leftrightarrow$	$F(\nu)\, G(\nu)$	$\displaystyle\sum_{k=-\infty}^{\infty} \delta(t - kT)$	$\leftrightarrow$	$\displaystyle\frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(\nu - \frac{k}{T}\right)$
$f(t)\, g(t)$	$\leftrightarrow$	$(F * G)(\nu)$	$\exp(-\pi t^2)$	$\leftrightarrow$	$\exp(-\pi\nu^2)$

DFT: $X[r] = \frac{1}{N} \sum_{k=0}^{N-1} x[k] \exp\left(-\frac{\mathrm{j}2\pi rk}{N}\right)$ IDFT: $x[k] = \sum_{r=0}^{N-1} X[r] \exp\left(\frac{\mathrm{j}2\pi rk}{N}\right)$

Hilbert transform: $\hat{f}(t) = f(t) * \frac{1}{\pi t}$

Convolution integral: $(f * h)(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t - \tau)\, \mathrm{d}\tau$

TURN OVER

1. (a) For a linear time invariant system *derive* an expression that relates the input $f(t)$, the output $g(t)$, and the impulse response function $h(t)$.

   (b) Calculate the Fourier transforms of the following functions:

   (i) $a(t) = t\,\mathrm{sinc}(4t)$

   (ii) $b(t) = \int_{-\infty}^{\infty} u(\tau)\exp(-\alpha\tau)\Pi(\beta(t-\tau))\,d\tau$ (where $\alpha > 0$)

   (iii) $c(t) = u(t)\cos(2\pi\nu_0 t)$

   (c) Starting from Parseval's theorem and using the properties of the Fourier transform on page 2, prove the convolution theorem.

   (d) For a real, finite energy signal $f(t)$, show that $f(t)$ and its Hilbert transform $\hat{f}(t)$ have the same energy spectral density.

   (e) Consider the sampling of a function with sampling period T . Which frequencies will be aliases of a given frequency ν_0? Show that $f(t) = \sin(2\pi\nu_0 t)$ and all of its aliases have the same value for their samples.

2. (a) The function $f(t)$ has Fourier transform $F(\nu)$ and is sampled at the times

$$t = 0,\ T,\ 2T,\ \ldots.\ ,\ (N-1)T$$

to give the vector of values

$$\{f[0],\ f[1],\ f[2],\ \ldots.\ ,\ f[N-1]\}.$$

(i) Consider the function

$$f_c(t) = \sum_{k=0}^{N-1} f[k]\,\delta(t-kT).$$

Show that

$$F[r] = \frac{1}{N} F_c\left(\frac{r}{NT}\right),$$

where the $F[r]$ are the elements of the discrete Fourier transform of the $f[k]$ and $F_c(\nu)$ is the Fourier transform of $f_c(t)$.

(ii) Using fact

$$f_c(t) = f(t)\,\Pi\left(\frac{t-(N-1)T/2}{NT}\right)\sum_{k=-\infty}^{\infty}\delta(t-kT)$$

derive an expression that relates $F_c(\nu)$ and $F(\nu)$. Using your expression, or otherwise, give the cause of aliasing, and of spectral leakage?

(b) For a vector $\mathbf{x}$ show that:

$$\text{IDFT}(\text{DFT}(\mathbf{x})) = \mathbf{x}.$$

(c) Consider the generalised function

$$f(t) = t\exp(j2\pi\nu_0 t).$$

(i) What is the Fourier transform of $f(t)$?

(ii) Show that your answer satisfies the formal definition for the Fourier transform of a generalised function.

3. (a) A stationary random process $x(t)$ has power spectral density given by

$$\Phi^p_{xx}(\nu) = A\,\mathrm{sinc}^2(\beta\nu).$$

(i) A large number of widely separated samples are taken of $x(t)$. What would you expect for the variance of this set?

(ii) How large does T need to be for $x(t)$ and $x(t+T)$ to be uncorrelated?

(b) A system has the impulse response function

$$h(t) = \frac{u(t)\cos(2\pi\nu_0 t)}{1+t^2}.$$

(i) Give a definition, in terms of the systems inputs and outputs, for what it means for a system to be *stable*.

(ii) State the stability theorem.

(iii) Is this system stable?

(iv) Prove your answer to (iii), starting from the definition of stability.

(c) A function $f(t)$ is sampled every T in time and the sampling satisfies the *Nyquist criterion*.

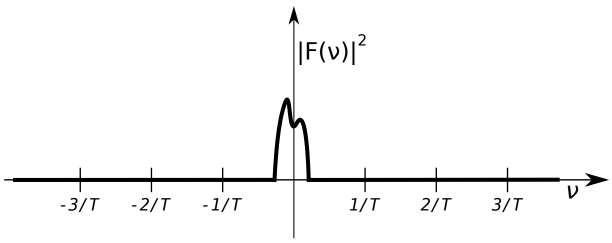
(i) What is the Nyquist criterion?

(ii) Derive an expression for $F_s(\nu)$, the Fourier transform of

$$f_s(t) = \sum_{k=-\infty}^{\infty} f(kT)\,\delta(t-kT).$$

Express your answer in terms of $F(\nu)$ the Fourier transform of $f(t)$.

(iii) Copy the graph below and add to it a sketch of $|F_s(\nu)|^2$.



(iv) Aided by your sketch derive an expression for $f(t)$ in terms of $f_s(t)$.

END

ELEC441

PLEASE DO NOT TURN OVER

THE EXAMINATION PAPER

UNTIL INSTRUCTED TO DO SO