UNIVERSITY OF OTAGO EXAMINATIONS 2017

PHYSICS

ELEC441

Linear Systems and Noise
Semester One

(TIME ALLOWED: 2 HOURS)

This examination paper comprises 5 pages

Candidates should answer questions as follows:

Answer TWO out of the THREE questions.
Questions carry equal weight.

The following material is provided:

Use of calculators:

No restriction on the model of calculator to be used, but no device with communication capability shall be accepted as a calculator.
(Subject to inspection by the examiners.)

Candidates are permitted copies of:

No additional material.

Other Instructions:

DO NOT USE RED INK OR PENCIL.
USEFUL RELATIONSHIPS can be found on page 2.

A table of Fourier transforms and properties

Forward: $F(\nu) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{j}2\pi\nu t} dt$ Inverse: $f(t) = \int_{-\infty}^{\infty} F(\nu)\, \mathrm{e}^{\mathrm{j}2\pi\nu t} d\nu$

Some properties

$$
\begin{aligned}
F(t) &\leftrightarrow f(-\nu) \\
f^*(t) &\leftrightarrow F^*(-\nu) \\
f(at) &\leftrightarrow \tfrac{1}{|a|} F\left(\tfrac{\nu}{a}\right) \\
f(t - t_0) &\leftrightarrow \mathrm{e}^{-\mathrm{j}2\pi\nu t_0} F(\nu) \\
\mathrm{e}^{\mathrm{j}2\pi\nu_0 t} f(t) &\leftrightarrow F(\nu - \nu_0) \\
\tfrac{d^n}{dt^n} f(t) &\leftrightarrow (\mathrm{j}2\pi\nu)^n F(\nu) \\
-\mathrm{j}2\pi t f(t) &\leftrightarrow \tfrac{dF(\nu)}{d\nu} \\
\int_{-\infty}^{t} f(\tau)\, d\tau &\leftrightarrow \tfrac{1}{\mathrm{j}2\pi\nu} F(\nu) + \tfrac{1}{2} F(0)\, \delta(\nu) \\
(f * g)(t) &\leftrightarrow F(\nu)\, G(\nu) \\
f(t)\, g(t) &\leftrightarrow (F * G)(\nu)
\end{aligned}
$$

Some transform pairs

$$
\begin{aligned}
\delta(t) &\leftrightarrow 1 \\
u(t)\, \mathrm{e}^{-at} &\leftrightarrow \tfrac{1}{\mathrm{j}2\pi\nu + a} \\
u(t) &\leftrightarrow \tfrac{1}{2}\delta(\nu) + \tfrac{1}{\mathrm{j}2\pi\nu} \\
\exp(\mathrm{j}2\pi\nu_0 t) &\leftrightarrow \delta(\nu - \nu_0) \\
\cos(2\pi\nu_0 t) &\leftrightarrow \tfrac{1}{2}\left[\delta(\nu - \nu_0) + \delta(\nu + \nu_0)\right] \\
\sin(2\pi\nu_0 t) &\leftrightarrow \tfrac{\mathrm{j}}{2}\left[-\delta(\nu - \nu_0) + \delta(\nu + \nu_0)\right] \\
\Pi(t) &\leftrightarrow \mathrm{sinc}(\nu) \\
\mathrm{sgn}(t) &\leftrightarrow \tfrac{1}{\mathrm{j}\pi\nu} \\
\sum_{k=-\infty}^{\infty} \delta(t - kT) &\leftrightarrow \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(\nu - \frac{k}{T}\right) \\
\exp(-\pi t^2) &\leftrightarrow \exp(-\pi\nu^2)
\end{aligned}
$$

DFT: $X[r] = \dfrac{1}{N} \sum_{k=0}^{N-1} x[k] \exp\left(-\dfrac{\mathrm{j}2\pi rk}{N}\right)$ IDFT: $x[k] = \sum_{r=0}^{N-1} X[r] \exp\left(\dfrac{\mathrm{j}2\pi rk}{N}\right)$

$$\text{Hilbert transform: } \hat{f}(t) = f(t) * \frac{1}{\pi t}$$

$$\text{Convolution integral: } (f * h)(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t - \tau)\, \mathrm{d}\tau$$

1. (a) Give definitions for:

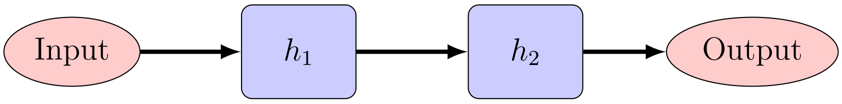
   (i) What it means for a system to be linear.

   (ii) What it means for a system to be shift invariant.

   (iii) The impulse response of a linear system.

   (iv) The transfer function for a linear shift invariant system.

   (b) Consider a linear time invariant system, h, which is formed by cascading two linear time invariant systems.



   (i) Derive an expression that relates the impulse response function of the cascaded system, $h(t)$, to the impulse response functions for the two sub-systems, $h_1(t)$ and $h_2(t)$.

   (ii) Derive an expression that relates the transfer function of the cascaded system, $H(\nu)$, with the transfer functions of the two sub-systems, $H_1(\nu)$ and $H_2(\nu)$.

   (c) For each of the following, sketch the functions and find their Fourier transforms.

   (i) $f(t) = \mathrm{sinc}(4t)$

   (ii) $g(t) = |\cos(t)|$

   (iii) $h(t) = \begin{cases} \cos^2(t), & t \in [-\pi/2, \pi/2] \\ 0, & \text{otherwise} \end{cases}$

2. (a) State and prove the convolution theorem.

(b) Using the formal definition of the Fourier transform of generalised functions show that the Fourier transform of

$$f(t) = \cos(2\pi\nu_0 t)$$

is

$$F(\nu) = \frac{\delta(\nu - \nu_0) + \delta(\nu + \nu_0)}{2}$$

(c) A system h, which produces an infinite number of non-diminishing echoes, has the impulse response

$$h(t) = \sum_{k=0}^{\infty} \delta(t - kT)$$

Is this system stable? Fully explain your answer.

(d) For the function $f(t) = \sin(8\pi t)\,\mathrm{sinc}(t/4)$:

(i) Calculate the Fourier transform of $f(t)$.

(ii) Sketch the $f(t)$ and its Fourier transform.

(iii) Calculate the analytic signal for $f(t)$.

(iv) Calculate the Hilbert transform of $f(t)$.

(e) Suppose the Fourier transform of $f(t)$ is $F(\nu)$. Derive an expression for the Fourier transform of $|f(t)|^2$.

3. (a) The function $f_p(t)$ is periodic with period T, that is $f_p(t) = f_p(t+T)$.

   (i) Show that the Fourier transform is of the form

$$F_p(\nu) = \sum_{k=-\infty}^{\infty} w_k \, \delta(\nu - k/T). \tag{1}$$

   (ii) Give an expression for the w_k in terms of $f_p(t)$.

(b) Consider the sampling of a function with sampling period T. Show that $f(t) = \sin(2\pi\nu_o t)$ and all of its aliases have the same value for their samples.

(c) White noise signal n with a power spectral density given by

$$\Phi_{nn}(\nu) = N_0/2, \tag{2}$$

is passed through a low pass filter with transfer function

$$H(\nu) = \frac{1}{j2\pi\nu + a}. \tag{3}$$

   (i) Show that

$$\exp(-a|t|) \leftrightarrow \frac{2a}{a^2 + 4\pi^2\nu^2}$$

   (ii) What is the power auto-correlation function and average power of the resulting signal?

   (iii) A set of samples, widely spaced in time is taken from the output. What would you expect for the standard deviation of this set of samples?

END

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THE EXAMINATION PAPER

UNTIL INSTRUCTED TO DO SO