UNIVERSITY OF OTAGO EXAMINATIONS 2016

PHYSICS

ELEC441

Linear Systems and Noise
Semester One

(TIME ALLOWED: 2 HOURS)

This examination paper comprises 5 pages

Candidates should answer questions as follows:

Answer TWO out of the THREE questions.
Questions carry equal weight.

The following material is provided:

Use of calculators:

No restriction on the model of calculator to be used, but no device with communication capability shall be accepted as a calculator.
(Subject to inspection by the examiners.)

Candidates are permitted copies of:

No additional material.

Other Instructions:

DO NOT USE RED INK OR PENCIL.
USEFUL RELATIONSHIPS can be found on page 2.

TURN OVER

A table of Fourier transforms and properties

Forward: $F(\nu) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{j}2\pi\nu t} dt$ Inverse: $f(t) = \int_{-\infty}^{\infty} F(\nu)\, \mathrm{e}^{\mathrm{j}2\pi\nu t} d\nu$

Some properties			Some transform pairs		
$F(t)$	$\leftrightarrow$	$f(-\nu)$	$\delta(t)$	$\leftrightarrow$	1
$f^*(t)$	$\leftrightarrow$	$F^*(-\nu)$	$u(t)\,\mathrm{e}^{-at}$	$\leftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu + a}$
$f(at)$	$\leftrightarrow$	$\frac{1}{\vert a\vert} F\left(\frac{\nu}{a}\right)$	$u(t)$	$\leftrightarrow$	$\frac{1}{2}\delta(\nu) + \frac{1}{\mathrm{j}2\pi\nu}$
$f(t - t_0)$	$\leftrightarrow$	$\mathrm{e}^{-\mathrm{j}2\pi\nu t_0} F(\nu)$	$\exp(\mathrm{j}2\pi\nu_0 t)$	$\leftrightarrow$	$\delta(\nu - \nu_0)$
$\mathrm{e}^{\mathrm{j}2\pi\nu_0 t} f(t)$	$\leftrightarrow$	$F(\nu - \nu_0)$	$\cos(2\pi\nu_0 t)$	$\leftrightarrow$	$\frac{1}{2}\left[\delta(\nu - \nu_0) + \delta(\nu + \nu_0)\right]$
$\frac{d^n}{dt^n} f(t)$	$\leftrightarrow$	$(\mathrm{j}2\pi\nu)^n F(\nu)$	$\sin(2\pi\nu_0 t)$	$\leftrightarrow$	$\frac{\mathrm{j}}{2}\left[-\delta(\nu - \nu_0) + \delta(\nu + \nu_0)\right]$
$-\mathrm{j}2\pi t f(t)$	$\leftrightarrow$	$\frac{dF(\nu)}{d\nu}$	$\Pi(t)$	$\leftrightarrow$	$\mathrm{sinc}(\nu)$
$\int_{-\infty}^{t} f(\tau)\, d\tau$	$\leftrightarrow$	$\frac{1}{\mathrm{j}2\pi\nu} F(\nu) + \frac{1}{2} F(0)\,\delta(\nu)$	$\mathrm{sgn}(t)$	$\leftrightarrow$	$\frac{1}{\mathrm{j}\pi\nu}$
$(f * g)(t)$	$\leftrightarrow$	$F(\nu)\, G(\nu)$	$\sum_{k=-\infty}^{\infty} \delta(t - kT)$	$\leftrightarrow$	$\frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(\nu - \frac{k}{T}\right)$
$f(t)\, g(t)$	$\leftrightarrow$	$(F * G)(\nu)$	$\exp(-\pi t^2)$	$\leftrightarrow$	$\exp(-\pi\nu^2)$

DFT: $X[r] = \frac{1}{N} \sum_{k=0}^{N-1} x[k] \exp\left(-\frac{\mathrm{j}2\pi rk}{N}\right)$ IDFT: $x[k] = \sum_{r=0}^{N-1} X[r] \exp\left(\frac{\mathrm{j}2\pi rk}{N}\right)$

Hilbert transform: $\hat{f}(t) = f(t) * \frac{1}{\pi t}$

Convolution integral: $(f * h)(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t - \tau)\, \mathrm{d}\tau$

TURN OVER

1. (a) Give definitions for each of the following:
   (i) a linear system;
   (ii) a shift invariant system;
   (iii) the impulse response for a linear time invariant system; and
   (iv) the transfer function for a linear time invariant system.

   (b) For a linear time invariant system *derive* an expression that relates the input $f(t)$, output $g(t)$, and impulse response function $h(t)$.

   (c) Show that $\exp(j2\pi\nu t)$ is an eigenfunction of a linear time invariant system. Derive an expression for the eigenvalue in terms of the impulse response.

   (d) (i) Prove the convolution theorem from the definition of the Fourier transform and the definition of the convolution.
   (ii) Show that the convolution is commutative, that is,

$$f * g = g * f.$$

   (iii) Show that

$$(f * g)' = f' * g = f * g'.$$

2. (a) Assuming that the Fourier transform of $f(t)$ is $F(\nu)$ and starting from the definition of the Fourier transform for ordinary functions, derive expressions, in terms of $F(\nu)$, for the Fourier transforms of

   (i) $f'(t)$

   (ii) $f(at)$

   (iii) $\mathrm{Re}[f(t)]$

   (iv) $f(t-b)$

   where $\mathrm{Re}[\cdot]$ denotes taking the real part and a and b are real constants.

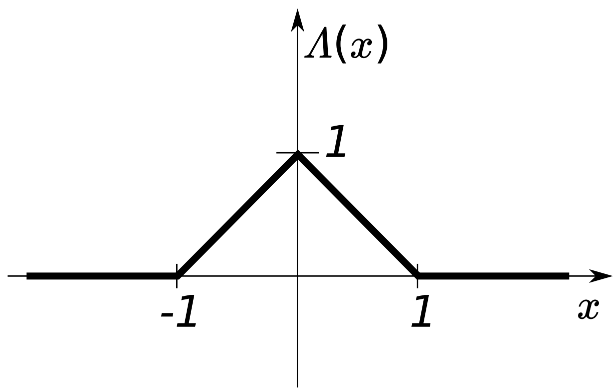
(b) $G(\nu)$ and $g(t)$ are generalised functions.

   (i) According to the definition of the Fourier transform of generalised functions, what requirements must be satisfied for the generalised function $G(\nu)$ to be the Fourier transform of the generalised function $g(t)$?

   (ii) Show from the definition above that

$$t \leftrightarrow \frac{j\delta'(\nu)}{2\pi}$$

(c) A Gaussian stationary random process $w(t)$ has spectrum $\Phi_{ww}(\nu) = A\,\Lambda(\nu T)$. (Here A and T are positive real constants and $\Lambda(\cdot)$ is the triangle function plotted below).



The resulting output signal is sampled at time values $\ldots -3T, -2T, -T, 0, T, 2T, 3T \ldots$.

   (i) Show that the resulting samples are independent random variables.

   (ii) What is their variance?

3. (a) A periodic function $f_p(t)$ can be created from a function, $f(t)$ via

$$f_p(t) = \sum_{k=-\infty}^{\infty} f(t - kT).$$

Show that $F_p(\nu)$, the Fourier transform of $f_p(t)$, consists of equally spaced delta functions. What is their spacing?

(b) A sequence of N numbers $\{x_0, x_1, x_2, ..., x_{N-1}\}$ is used to create the following generalised function

$$b(t) = \sum_{k=-\infty}^{\infty} x_{k \bmod N}\, \delta(t - kT).$$

Show that $B(\nu)$, the Fourier transform of $b(t)$, is given by

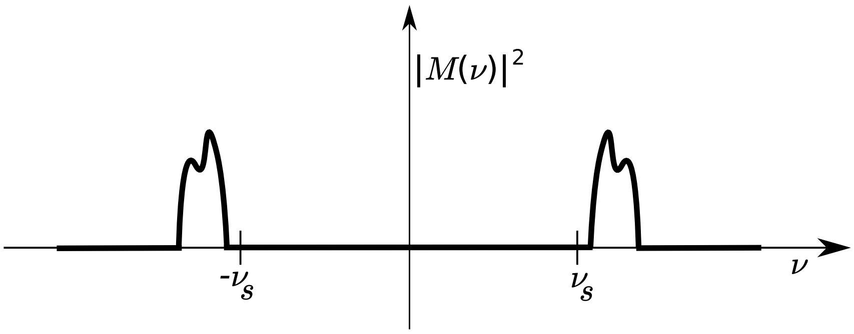
$$B(\nu) = \frac{1}{T} \sum_{r=-\infty}^{\infty} X_{r \bmod N}\; \delta\left(\nu - \frac{r}{NT}\right),$$

where $\{X_0, X_1, X_2, ..., X_{N-1}\}$ is the discrete Fourier transform of $\{x_0, x_1, x_2, ..., x_{N-1}\}$.

(c) A real valued function $m(t)$ is sampled with sampling period $\nu_s \equiv 1/T$ resulting in

$$m_s(t) = \sum_{k=-\infty}^{\infty} m(kT)\delta(t - kT).$$

As shown below, $M(\nu)$, the Fourier transform of $m(t)$, is only non zero for frequencies given by $\nu_s < |\nu| < \frac{3}{2}\nu_s$



(i) Derive an expression for $M_s(\nu)$, the Fourier transform of $m_s(t)$ in terms of $M(\nu)$.

(ii) Copy the above graph into your answer booklet and add to it a sketch of $|M_s(\nu)|^2$

(iii) Sketch on your diagram and give the formula for a function that when multiplied by $M_s(\nu)$ gives the result $M(\nu)$. *(Hint: This function is the sum of two scaled and shifted top-hat functions.)*

(iv) Even though the sampling in this situation doesn't fit the standard sampling theorem criterion, $m(t)$ can still be recovered from its samples. With the aid of a graph explain how this is done and give a formula for $m(t)$ in terms of the samples $m(kT)$.

END